Fields Institute Communications
Volume **00**, 0000

NEW IDEAS for RESOLUTION of SINGULARITIES in ARBITRARY CHARACTERISTIC

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Dedicated to Professor Takuo Fukuda on his sixtieth birthday

Abstract. I have succeeded in showing that any two-dimensional hypersurface singularities of germs of varieties in any characteristic can be resolved by iterated monoidal transformations with centers in smooth subvarieties. The new proof for the two-dimensional case depends on new ideas. Ideas are essentially different from Abhyankar's one in [1] and Lipman's one in [5]. It seems to be possible to generalize the new proof into higher dimensional cases, if we add several ideas further. In this article I try to explain my new ideas rather than the partial result I explained at the conference.

1 Introduction

At the conference I explained my partial result claiming that any hypersurface singularities of germs of varieties in positive characteristic can be resolved by iterated monoidal transformations with centers in smooth subvarieties, if we have a valuation ring of iterated divisor type associated with the germ. However, the existence of a certain valuation ring may be a strong condition, and there were several difficulties to remove this condition.

After coming back to Japan, through intensive study, I have succeeded in showing that any two-dimensional hypersurface singularities of germs of varieties in any characteristic can be resolved by iterated monoidal transformations with centers in smooth subvarieties. The new proof for the two-dimensional case depends on new ideas. Ideas are essentially different from Abhyankar's one in [1] and Lipman's one in [5]. I got new ideas while I was checking Section eight of Abhyankar [1]. Section eight of [1] contains a lot of ideas, but it has nothing to do with the main result in [1]. Adding my own idea of a coordinate change into special coordinates, I achieved

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2000 *Mathematics Subject Classification.* Primary 14E15; Secondary 32S45, 13F25.

Key words and phrases. hypersurface, singularity, resolution of singularities, monoidal transformation, blowing-up, normal crossing, Weierstrass preparation theorem.

a new proof. The origin of my idea of a special coordinate change is the concept of "reductive" in Urabe [8], [9].

It seems to be possible to generalize the new proof into higher dimensional cases, if we add several ideas further. I am now working in this generalization. I need much time.

In this article I try to explain my new ideas rather than the partial result I explained at the conference.

Throughout this article we fix the ground field k. We assume that k is an infinite field.

Let Γ be a totally ordered abelian group. Often in this article we associate a special element ∞ called the *infinity* with Γ. We assume that the element ∞ satisfies the following two conditions:

1. $\infty \geq \infty$, $\infty + \infty = \infty$, and $n\infty = \infty n = \infty$ for every positive integer n.
2. For every element $\gamma \in \Gamma$ $\gamma \neq \infty$, $\gamma < \infty$ and $\gamma + \infty = \infty + \gamma = \infty$.

By $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ we denote the ring of integers, the rational number field and the real number field respectively. By $\mathbb{Z}_0$, $\mathbb{Q}_0$ and $\mathbb{R}_0$ we denote the set of non-negative integers, the set of non-negative rational numbers and the set of non-negative real numbers respectively.

The number of elements in a finite set X is denoted by $\sharp X$.

2 Ring theory

In this section we review several important points in the ring theory and explain non-standard concepts.

By a ring we mean a commutative ring with the identity 1. We use standard terminologies in the commutative ring theory. (Matsumura [6]. Zariski-Samuel [10].)

Let A be a ring. By A^* we denote the set of invertible elements of A, i.e.,

$$A^* = \{a \in A \mid \text{ There exists an element } b \in A \text{ with } ab = 1\}.$$

A ring with the unique maximal ideal is called a *local ring*. For a local ring A by $M(A)$ we denote the maximal ideal of A. A local ring A is the disjoint union of A^* and $M(A)$.

Let n be a non-negative integer, and $x_1, x_2, \ldots, x_n$ n of variables. We consider the formal power series ring $k[[x]] = k[[x_1, x_2, \ldots, x_n]]$ over k. Let $P = (x_1, x_2, \ldots, x_n)$ be an ordered parameter system of $k[[x]]$. An element $Q = (y_1, y_2, \ldots, y_n) \in k[[x]]^n$ such that the set $\{y_1, y_2, \ldots, y_n\}$ is a parameter system of the ring $k[[x]]$ is called an *ordered parameter system* of $k[[x]]$. Any element $f \in k[[x]]$ can be written uniquely in the form

$$f = \sum_{\alpha \in \mathbb{Z}_0^n} \mathrm{coeff}(f, P, \alpha) x^\alpha,$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}_0^n$, $\alpha_1, \alpha_2, \ldots, \alpha_n$ are non-negative integers, $\mathrm{coeff}(f, P, \alpha) \in k$ for any $\alpha \in \mathbb{Z}_0^n$, and $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$.

The set of exponents of f with respect to P is denoted by $\mathrm{ex}(f, P)$.

$$\mathrm{ex}(f, P) = \{\alpha \in \mathbb{Z}_0^n \mid \mathrm{coeff}(f, P, \alpha) \neq 0\}.$$

Definition 2.1 1. Let $P = (x_1, x_2, \ldots, x_n)$ be an ordered parameter system of the ring $k[[x]]$ and $f \in k[[x]]$. We say that f has a *normal crossing* *with respect to* P, if there are non-negative integers $\beta_1, \beta_2, \ldots, \beta_n$ with

$$\frac{f}{x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n}} \in k[[x]]^*.$$

2. An element $f \in k[[x]]$ is said to have a *normal crossing*, if there is an ordered parameter system P of $k[[x]]$ such that f has a normal crossing with respect to P.

For any $A = (A_1, A_2, \ldots, A_n)$, $B = (B_1, B_2, \ldots, B_n) \in \mathbb{R}^n$, we denote $(A, B) = \sum_{i=1}^n A_i B_i$.

Let $w = (w_1, w_2, \ldots, w_n) \in \mathbb{Z}_0^n$, and $P = (x_1, x_2, \ldots, x_n)$ an ordered parameter system of $k[[x]]$. We will define a map

$$\mathrm{ord}_w^P : k[[x]] \to \mathbb{Z}_0 \cup \{\infty\}$$

associated with w and P. The value $\mathrm{ord}_w^P(f) \in \mathbb{Z}_0 \cup \{\infty\}$ for $f \in k[[x]]$ is called the *order* of f with *weight* w with respect to P. Let $f \in k[[x]]$ be an element. If $f = 0$, then we put $\mathrm{ord}_w^P(f) = \infty$. Assume $f \neq 0$. Since $f \neq 0$, the set $\mathrm{ex}(f, P)$ is not empty. We define

$$\mathrm{ord}_w^P(f) = \min\{(w, \alpha) \mid \alpha \in \mathrm{ex}(f, P)\} \in \mathbb{Z}_0.$$

Lemma 2.2 1. *If* $f, g \in k[[x]]$, *then* $\mathrm{ord}_w^P(fg) = \mathrm{ord}_w^P(f) + \mathrm{ord}_w^P(g)$.
2. *If* $f, g \in k[[x]]$, *then* $\mathrm{ord}_w^P(f + g) \geq \min\{\mathrm{ord}_w^P(f), \mathrm{ord}_w^P(g)\}$.

Let E denote the field of fractions of $k[[x]]$. We can define an extended map

$$\mathrm{ord}_w^P : E \to \mathbb{Z} \cup \{\infty\}$$

by putting $\mathrm{ord}_w^P(h) = \mathrm{ord}_w^P(f) - \mathrm{ord}_w^P(g)$, where $h \in E$, and $f, g \in k[[x]]$ are elements satisfying $h = f/g$ and $g \neq 0$.

Lemma 2.3 1. *The map* $\mathrm{ord}_w^P : E \to \mathbb{Z} \cup \{\infty\}$ *associated with an element* $w \in \mathbb{Z}_0^n$ *and an ordered parameter system* P *is a valuation of the field* E *with the value group* $\mathbb{Z}$.
2. $\mathrm{ord}_w^P(f) \geq 0$ *for every* $f \in k[[x]]$.
3. *Let* m *be an integer with* $0 \leq m \leq n$. *Assume* $w_i = 1$ *for every* $1 \leq i \leq m$, *and* $w_i = 0$ *for every* $m < i \leq n$. *Let* $Q = (y_1, y_2, \ldots, y_n)$ *be another ordered parameter system of* $k[[x]]$. *If the ideal generated by* $x_1, x_2, \ldots, x_m$ *and the ideal generated by* $y_1, y_2, \ldots, y_m$ *coincide, then we have* $\mathrm{ord}_w^P = \mathrm{ord}_w^Q$.

When $w_i = 0$ or 1 for every $1 \leq i \leq n$ and $w_i = 1$ for some $1 \leq i \leq n$, we write

$$\mathrm{ord}_w^P = \mathrm{ord}_{x_{i_1}, x_{i_2}, \ldots, x_{i_m}} = \mathrm{ord}_{\mathfrak{I}},$$

where $i_1, i_2, \ldots, i_m$ are mutually different integers such that $\{i_1, i_2, \ldots, i_m\} = \{i \in \mathbb{Z}_0 \mid 1 \leq i \leq n, w_i = 1\}$, and $\mathfrak{I}$ denotes the ideal generated by $x_{i_1}, x_{i_2}, \ldots, x_{i_m}$. Moreover, if $\mathfrak{I} = M(k[[x]])$ is the maximal ideal, then we write $\mathrm{ord} = \mathrm{ord}_{\mathfrak{I}}$. The value $\mathrm{ord}(f)$ is called the *order* of f.

Let $f = \sum_{\alpha \in \mathbb{Z}_0^n} \mathrm{coeff}(f, P, \alpha) x^\alpha \in k[[x]]$ be a non-zero element. Let $T = \{\alpha \in \mathrm{ex}(f, P) \mid (w, \alpha) = \mathrm{ord}_w^P(f)\}$. Putting

$$\mathrm{in}_w^P(f) = \sum_{\alpha \in T} \mathrm{coeff}(f, P, \alpha) x^\alpha,$$

we call $\mathrm{in}_w^P(f)$ the *w-initial polynomial* of f with respect to P. For $f = 0 \in k[[x]]$ we define $\mathrm{in}_w^P(0) = 0$.

We can check that $\mathrm{in}_w^P(fg) = \mathrm{in}_w^P(f)\mathrm{in}_w^P(g)$ for any $f, g \in k[[x]]$.

Assume that $w_i = 0$ for $1 \le i \le m$ and $w_i > 0$ for $m < i \le n$. One sees that $\mathrm{in}_w^P(f) \in k[[x_1, x_2, \ldots, x_m]][x_{m+1}, x_{m+2}, \ldots, x_n]$.

We say that f is a *w-homogeneous polynomial* with respect to P, if $f = \mathrm{in}_w^P(f)$. If $w_i > 0$ for every $1 \le i \le n$, then any w-homogeneous polynomial contains only a finite number of terms, and it is a polynomial with coefficients in k.

When $w_i = 0$ or 1 for every $1 \le i \le n$ and $w_i = 1$ for some $1 \le i \le n$, we write

$$\mathrm{in}_w^P = \mathrm{in}_{x_{i_1}, x_{i_2}, \ldots, x_{i_m}}^P,$$

where $i_1, i_2, \ldots, i_m$ are mutually different integers such that $\{i_1, i_2, \ldots, i_m\} = \{i \in \mathbb{Z}_0 \mid 1 \le i \le n, w_i = 1\}$. In the case where $w_i = 1$ for every $1 \le i \le n$, we write $\mathrm{in}_{x_1, x_2, \ldots, x_n}^P = \mathrm{in}^P$ and a w-homogeneous polynomial is called a *homogeneous polynomial*.

Let $e_1, e_2, \ldots, e_n$ denote the standard basis of $\mathbb{R}^n$. We consider a subset S of $\mathbb{R}^n$. The set

$$I_S = \{i \in \mathbb{Z}_0 \mid 1 \le i \le n, A + re_i \in S \text{ for any } A \in S \text{ and for any } r \in \mathbb{R}_0.\}$$

is called the *set of infinite directions* of S.

Lemma 2.4 *Let M be a positive integer, and T a subset of $(\frac{1}{M}\mathbb{Z}_0)^n$. Let S denote the minimum convex subset of $\mathbb{R}^n$ containing $T + \mathbb{Z}_0^n$.*

1. *The subset S is a convex polyhedron with a finite number of faces.*
2. *$I_S = \{1, 2, \ldots, n\}$.*
3. *Any vertex of S belongs to $(\frac{1}{M}\mathbb{Z}_0)^n$.*
4. *Assume $S \ne \emptyset$. For every $w = (w_1, w_2, \ldots, w_n) \in \mathbb{Z}_0^n$ the set $\Delta(w, S) = \{A \in S \mid (w, A) = \min\{(w, B) \mid B \in S\}\}$ is a face of S.*
5. *For any face F of S, there is an element $w \in \mathbb{Z}_0^n$ with $F = \Delta(w, S)$.*
6. *Assume $S \ne \emptyset$. Let $w = (w_1, w_2, \ldots, w_n) \in \mathbb{Z}_0^n$ and $F = \Delta(w, S)$. We have $w_i = 0$, if and only if, $i \in I_F$.*
7. *A face F of S is compact, if and only if, $I_F = \emptyset$.*

Let $P = (x_1, x_2, \ldots, x_n)$ be an ordered parameter system of $k[[x]]$, and $f \in k[[x]]$. The minimum convex subset of $\mathbb{R}^n$ containing $\mathrm{ex}(f, P) + \mathbb{Z}_0^n$ is called the *Newton polyhedron* of f, and is denoted by $\Gamma_+(f, P)$. For any subset T of $\mathbb{R}^n$

$$\mathrm{ps}_T^P(f) = \begin{cases} \sum_{\alpha \in \mathrm{ex}(f,P) \cap T} \mathrm{coeff}(f, P, \alpha) x^\alpha \in k[[x]] & \text{if } \mathrm{ex}(f, P) \cap T \ne \emptyset, \\ 0 \in k[[x]] & \text{if } \mathrm{ex}(f, P) \cap T = \emptyset. \end{cases}$$

is called the *partial sum* of f over T with respect to P.

Lemma 2.5 1. *$\Gamma_+(f, P) = \emptyset$ if and only if $f = 0$.*
2. *Assume $f \ne 0$. For every $w = (w_1, w_2, \ldots, w_n) \in \mathbb{Z}_0^n$,* $\mathrm{ord}_w^P(f) = \min\{(w, A) \mid A \in \Gamma_+(f, P)\}$.
3. *Assume that a subset $T \subset \mathbb{R}^n$ and an element $w = (w_1, w_2, \ldots, w_n) \in \mathbb{Z}_0^n$ satisfies $T \cap \Gamma_+(f, P) = \Delta(w, \Gamma_+(f, P))$, if $f \ne 0$. We have $\mathrm{in}_w^P(f) = \mathrm{ps}_T^P(f)$.*
4. *Let T be a subset of $\mathbb{R}^n$, and $a_1, a_2, \cdots, \in k$, $f_1, f_2, \cdots \in k[[x]]$ elements such that the sum $\sum a_i f_i$ exists in $k[[x]]$. We have $\mathrm{ps}_T^P(\sum a_i f_i) = \sum a_i \mathrm{ps}_T^P(f_i)$.*
5. *Let Γ be a convex subset of $\mathbb{R}^n$ such that $\mathrm{ex}(f, P) \subset \Gamma$, and F a face of Γ. For every positive integer j we have $\mathrm{ps}_{jF}^P(f^j) = \mathrm{ps}_F^P(f)^j$.*
6. *Let Γ be a convex subset of $\mathbb{R}^n$, F a face of Γ, $f, g \in k[[x]]$, and $r, s \in \mathbb{R}_0$. Assume that $\mathrm{ex}(f, P) \subset r\Gamma$ and $\mathrm{ex}(g, P) \subset s\Gamma$. We have $\mathrm{ps}_{(r+s)F}^P(fg) = \mathrm{ps}_{rF}^P(f)\mathrm{ps}_{sF}^P(g)$.*

Theorem 2.6 (Weierstrass division theorem) *Assume* $n > 0$. *Let* $P = (x_1, x_2, \dots, x_n)$ *be an ordered parameter system of the ring* $k[[x]]$, *and* $w = (w_1, w_2, \dots, w_n) \in \mathbb{Z}_0^n$. *We assume that* $w_i > 0$ *for every* $1 \le i \le n-1$ *and* $w_n = 1$. *We write* $k[[x']] = k[[x_1, x_2, \dots, x_{n-1}]]$. *Let* $f \in k[[x]]$. *Assume that* $\mathrm{in}_w^P(f)(0, 0, \dots, 0, 1) \neq 0$, *where* $\mathrm{in}_w^P(f)(0, 0, \dots, 0, 1)$ *denotes the value of the w-initial polynomial of f at* $x_1 = x_2 = \cdots = x_{n-1} = 0$, $x_n = 1$. *Let* $L = \mathrm{ord}_w^P(f)$.

For every $h \in k[[x]]$ *there exists a unique set of an element* $e \in k[[x]]$ *and L numbered elements* $H_1, H_2, \dots, H_L \in k[[x']]$ *satisfying*

$$h = ef + \sum_{j=1}^{L} H_j x_n^{L-j}.$$

Corollary 2.7 (Weierstrass preparation theorem) *We assume the same assumptions as in the above theorem.*

1. *There exists a unique set of an element* $e \in k[[x]]$ *and L numbered elements* $G_1, G_2, \dots, G_L \in k[[x']]$ *satisfying*

$$ef = x_n^L + \sum_{j=1}^{L} G_j x_n^{L-j}.$$

2. $e \in k[[x]]^*$, *and* $\mathrm{ord}_w^P(G_j) \ge j$ *for every* $1 \le j \le L$.

Proof It follows from the special case of Theorem 2.6 where $h = x_n^L$. □

Lemma 2.8 *Let* $f \in k[[x]]$ *be a non-zero homogeneous polynomial with respect to an ordered parameter system* $P = (x_1, x_2, \dots, x_n)$.

1. *There exist* $n-1$ *elements* $a_1, a_2, \dots, a_{n-1} \in k$ *with* $f(a_1, a_2, \cdots, a_{n-1}, 1) \neq 0$, *where* $f(a_1, a_2, \cdots, a_{n-1}, 1)$ *denotes the value of f at* $x_1 = a_1, x_2 = a_2, \dots, x_{n-1} = a_{n-1}, x_n = 1$.
2. *Assume that* $f(a_1, a_2, \cdots, a_{n-1}, 1) \neq 0$ *for* $a_1, a_2, \dots, a_{n-1} \in k$. *Let* $y_i = x_i - a_i x_n$ *for* $1 \le i \le n-1$, *and* $y_n = x_n$. *Then* $Q = (y_1, y_2, \dots, y_n)$ *is an ordered parameter system of* $k[[x]]$, *and the value of f at* $y_1 = y_2 = \cdots = y_{n-1} = 0, y_n = 1$ *is not zero.*

Remark Recall that we have assumed that k is an infinite field. Lemma 2.8 is the essential reason why it is difficult to treat the case of a finite field in the resolution theory.

Lemma 2.9 *Let E be a field,* $v : E \to \Gamma \cup \{\infty\}$ *a valuation, t a transcendental element over E, L a positive integer, and* $G_1, G_2, \dots, G_L, H_1, H_2, \dots, H_L \in E$ *2L elements satisfying*

$$t^L + \sum_{j=1}^{L} G_j t^{L-j} = \prod_{j=1}^{L} (t + H_j).$$

We have

$$\min\{\frac{1}{j} v(G_j) \mid 1 \le j \le L\} = \min\{v(H_j) \mid 1 \le j \le L\}.$$

Let $P = (x_1,\ x_2, \dots, x_n)$ be an ordered parameter system of $k[[x]]$, $w = (w_1, w_2, \dots, w_n) \in \mathbb{Z}_0^n$, L an positive integer, and $G = (G_1, G_2, \dots, G_L) \in k[[x]]^L$. We define

$$\mathrm{ord}_w^P(G) = \min\{\frac{1}{j} \mathrm{ord}_w^P(G_j) \mid 1 \le j \le L\}.$$

The value $\mathrm{ord}_w^P(G) \in \mathbb{Z}_0 \cup \{\infty\}$ for $G \in k[[x]]^L$ is called the *order* of G with *weight* w with respect to P. Let

$$\bar{G}_j = \begin{cases} \mathrm{in}_w^P(G_j) & \text{if } \mathrm{ord}_w^P(G_j) = j\mathrm{ord}_w^P(G), \\ 0 & \text{if } \mathrm{ord}_w^P(G_j) > j\mathrm{ord}_w^P(G). \end{cases}$$

We define $\mathrm{in}_w^P(G) = (\bar{G}_1, \bar{G}_2, \ldots, \bar{G}_L)$. We call $\mathrm{in}_w^P(G)$ the *w-initial polynomial* of G with respect to P.

The minimum convex subset of $\mathbb{R}^n$ containing $\left(\bigcup_{j=1}^L (1/j)\mathrm{ex}(G_j, P)\right) + \mathbb{Z}_0^n$ is called the *Newton polyhedron* of G, and is denoted by $\Gamma_+(G, P)$. For any subset T of $\mathbb{R}^n$

$$\mathrm{ps}_T^P(G) = (\mathrm{ps}_T^P(G_1), \mathrm{ps}_{2T}^P(G_2), \ldots, \mathrm{ps}_{jT}^P(G_j), \ldots, \mathrm{ps}_{LT}^P(G_L)) \in k[[x]]^L$$

is called the *partial sum* of G over T with respect to P.

Lemma 2.10 1. *$\Gamma_+(G, P) = \emptyset$ if and only if $G = 0$.*

2. *For every integer j with $1 \le j \le L$ we have $\Gamma_+(G_j, P) \subset j\Gamma_+(G, P)$.*

3. *Assume $G \neq 0$. For every $w = (w_1, w_1, \ldots, w_n) \in \mathbb{Z}_0^n$, $\mathrm{ord}_w^P(G) = \min\{(w, A) \mid A \in \Gamma_+(G, P)\}$.*

4. *Assume that a subset $T \subset \mathbb{R}^n$ and an element $w = (w_1, w_1, \ldots, w_n) \in \mathbb{Z}_0^n$ satisfies $T \cap \Gamma_+(G, P) = \Delta(w, \Gamma_+(G, P))$, if $G \neq 0$. We have $\mathrm{in}_w^P(G) = \mathrm{ps}_T^P(G)$.*

3 Essential definitions

Let n be a non-negative integer, and $x_1, x_2, \ldots, x_n$ n of variables. We consider the formal power series ring $k[[x]] = k[[x_1, x_2, \ldots, x_n]]$ over k in the former half of this section. Let $P = (x_1, x_2, \ldots, x_n)$ be an ordered parameter system of $k[[x]]$, x_{n+1} a transcendental element over $k[[x]]$, L an positive integer, and $G = (G_1, G_2, \ldots, G_L) \in k[[x]]^L$ an element.

Definition 3.1 Let Δ be a face of the Newton polyhedron $\Gamma_+(G, P)$. We say that Δ is *first erasable*, if there exists an element $E \in k[[x]]$ satisfying

$$x_{n+1}^L + \sum_{j=1}^L \mathrm{ps}_{j\Delta}^P(G_j) x_{n+1}^{L-j} = (x_{n+1} + E)^L. \tag{3.1}$$

Lemma 3.2 1. *For every $E \in k[[x]]$ there exists a unique element $\hat{G} = (\hat{G}_1, \hat{G}_2, \ldots, \hat{G}_L) \in k[[x]]^L$ satisfying*

$$x_{n+1}^L + \sum_{j=1}^L G_j x_{n+1}^{L-j} = (x_{n+1} + E)^L + \sum_{j=1}^L \hat{G}_j (x_{n+1} + E)^{L-j}. \tag{3.2}$$

2. *Let $G_0 = \hat{G}_0 = 1$. We consider equalities below.*

$$G_j = \sum_{i=0}^j \binom{L-i}{L-j} \hat{G}_i E^{j-i}, \tag{3.3}$$

$$\hat{G}_j = \sum_{i=0}^j \binom{L-i}{L-j} G_i (-E)^{j-i}, \tag{3.4}$$

The following three conditions are equivalent:

(a) *The equality 3.2 holds.*

  - (b) *The equality 3.3 holds for every* $1 \leq j \leq L$.
  - (c) *The equality 3.4 holds for every* $1 \leq j \leq L$.

Theorem 3.3

1. *Let* Δ *be a first erasable face of the Newton polyhedron* $\Gamma_+(G, P)$*, and* Δ_1 *a face with* $\Delta_1 \subset \Delta$*. Then, the face* Δ_1 *is first erasable.*
2. *There is an element* $E \in k[[x]]$ *satisfying three conditions below. Let* $\hat{G} \in k[[x]]^L$ *be the element satisfying the relation 3.2 in Lemma 3.2.*
    - (a) *Every face of* $\Gamma_+(\hat{G}, P)$ *is not first erasable.*
    - (b) $\Gamma_+(\hat{G}, P) \subset \Gamma_+(G, P)$.
    - (c) $\mathrm{ex}(E, P) \subset \Gamma_+(G, P) - \Gamma_+(\hat{G}, P)$
3. *Let* $E \in k[[x]]$ *be an arbitrary element, and* $\hat{G} \in k[[x]]^L$ *the element satisfying the relation 3.2 in Lemma 3.2. Assume that every face of* $\Gamma_+(\hat{G}, P)$ *is not first erasable.*
    - (a) $\Gamma_+(E, P) \subset \Gamma_+(G, P)$, $\Gamma_+(\hat{G}, P) \subset \Gamma_+(G, P)$
    - (b) *If a face* Δ *of* $\Gamma_+(G, P)$ *is not first erasable, then* $\Delta \cap \Gamma_+(\hat{G}, P) \neq \emptyset$.
    - (c) *If* $\Gamma_+(\hat{G}, P) = \Gamma_+(G, P)$*, then* $\Gamma_+(G, P)$ *has no first erasable face.*

Definition 3.4 Let M be a positive integer, T a subset of $(\frac{1}{M}\mathbb{Z}_0)^n$, and S the minimum convex subset of $\mathbb{R}^n$ containing $T + \mathbb{Z}_0^n$.

1. We say that S is *of Weierstrass type*, if $n > 0$ and the convex polyhedron $\pi(S)$ in $\mathbb{R}^{n-1}$ has only one vertex, where $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ denotes the projection satisfying $\pi(A_1, A_2, \ldots, A_{n-1}, A_n) = (A_1, A_2, \ldots, A_{n-1})$.

   We assume below that S is of Weierstrass type.
2. Let A' be the unique vertex of $\pi(S)$. There exists a unique vertex A of S with $A' = \pi(A)$. We call A the *characteristic vertex* of S.
3. We write $A = (A', A_n)$, where $A' \in \mathbb{R}^{n-1}$ and $A_n \in \mathbb{R}$. The domain $D = \mathbb{R}^{n-1} \times \{r \in \mathbb{R} \mid r < A_n\}$ is called the *characteristic domain* of S.
4. The map $\rho : D \to \mathbb{R}^{n-1}$ from D to $\mathbb{R}^{n-1}$ defined by
$$\rho(B) = \frac{1}{A_n - B_n}(B' - A')$$
   is called the *characteristic map* of S, where $B = (B', B_n)$, $B' \in \mathbb{R}^{n-1}$, $B_n \in \mathbb{R}$, and $B_n < A_n$.

Remark When $n = 1$ or 2, S is of Weierstrass type, if and only if, S is not empty.

Lemma 3.5 *Let* M *be a positive integer and* T *a subset of* $(\frac{1}{M}\mathbb{Z}_0)^n$*. Assume that the minimum convex subset* S *of* $\mathbb{R}^n$ *containing* $T + \mathbb{Z}_0^n$ *is of Weierstrass type. Let* $A = (A', A_n)$*,* D*, and* $\rho : D \to \mathbb{R}^{n-1}$ *denote the characteristic vertex, the characteristic domain, and the characteristic map of* S *respectively. Let* V *be the set of vertices of* S*.*

1. $V - \{A\} \subset D$.
2. $S' = \rho(S \cap D)$ *coincides with the minimum convex subset of* $\mathbb{R}^{n-1}$ *containing* $\rho(V - \{A\}) + \mathbb{Z}_0^{n-1}$.
3. $S' = \emptyset$*, if and only if,* S *has only one vertex.*
4. *Let* M' *be the least common multiple of* $1, 2, \ldots, MA_n$. $\rho(V - \{A\}) \subset (\frac{1}{M'}\mathbb{Z}_0)^{n-1}$.
5. *Let* F *be a face of* S *with* $A \in F$ *and* $F \cap D \neq \emptyset$*. Then,* $F' = \rho(F \cap D)$ *is a face of* S'*. Moreover, the following holds:*
    - (a) $\dim F = \dim F' + 1$.

(b) $I_F \cap \{1, 2, \dots, n-1\} = I_{F'}$.
(c) *The following two conditions are equivalent:*
  (i) $n \in I_F$.
  (ii) *If* $B' = (B_1, B_2, \dots, B_{n-1}) \in F'$, *and if* $i \in \{1, 2, \dots, n-1\} - I_{F'}$, *then* $B_i = 0$.
(d) *The following conditions are equivalent:*
  (i) F *is compact.*
  (ii) $F - \{A\} \subset D$.
  (iii) F' *is compact.*
  (iv) $F = (\rho^{-1}(F') \cap S) \cup \{A\}$.

6. *For every face* F' *of* S' *there exists a unique face* F *of* S *satisfying* $A \in F$, $F \cap D \neq \emptyset$ *and* $\rho(F \cap D) = F'$.

In what follows, we write $k[[x']] = k[[x_1, x_2, \dots, x_{n-1}]]$, $P' = (x_1, x_2, \dots, x_{n-1})$, and $x'^{\alpha'} = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_{n-1}^{\alpha_{n-1}}$ for $\alpha' = (\alpha_1, \alpha_2, \dots, \alpha_{n-1}) \in \mathbb{Z}_0^{n-1}$.

For any ordered parameter system $Q = (y_1, y_2, \dots, y_n)$ of $k[[x]]$, $H \in k[[x]]$ and $\alpha \in \mathbb{R}^n - \mathbb{Z}_0^n$, we define $\mathrm{coeff}(H, Q, \alpha) = 0$.

For a point $B \in \mathbb{R}^n$ and a subset $T \subset \mathbb{R}^n$ we put

$$\mathrm{Cone}(B, T) = \{B + r(C - B) \mid C \in T, r \in \mathbb{R}_0\},$$

and call it the *cone* over T with vertex B.

Definition 3.6 Assume that the Newton polyhedron $\Gamma_+(G, P)$ is of Weierstrass type. Let $A = (A', A_n)$, D, and $\rho : D \to \mathbb{R}^{n-1}$ denote the characteristic vertex, the characteristic domain, and the characteristic map of $\Gamma_+(G, P)$ respectively.

We say that a face Δ of $\Gamma_+(G, P)$ is *second erasable*, if $A \in \Delta$ and $\Delta \cap D \neq \emptyset$, and if there exist elements $E \in k[[x]]$ and $F \in k[[x']]$ satisfying the following two conditions. The corresponding face of $\Gamma_+(G, P)' = \rho(\Gamma_+(G, P) \cap D)$ is denoted by $\Delta' = \rho(\Delta \cap D)$:

1. $\mathrm{ex}(E, P) \subset \mathrm{Cone}(A, \Delta) - \{A\}$, $\mathrm{ex}(F, P') \subset \Delta'$
2.

$$x_{n+1}^L + \sum_{j=1}^{L} \mathrm{ps}_{j\Delta}^P(G_j) x_{n+1}^{L-j} = \tag{3.5}$$

$$(x_{n+1} + E)^L + \sum_{j=1}^{L} \mathrm{coeff}(G_j, P, jA) x'^{jA'} (x_n + F)^{jA_n} (x_{n+1} + E)^{L-j}.$$

A face Δ' of $\Gamma_+(G, P)'$ is said to be *second erasable*, if the corresponding face Δ of $\Gamma_+(G, P)$ is second erasable.

Theorem 3.7 *Assume that the Newton polyhedron* $\Gamma_+(G, P)$ *is of Weierstrass type. Let* $A = (A', A_n)$, D, *and* $\rho : D \to \mathbb{R}^{n-1}$ *denote the characteristic vertex, the characteristic domain, and the characteristic map of* $\Gamma_+(G, P)$ *respectively.*

1. *Let* Δ *and* Δ_1 *be faces of* $\Gamma_+(G, P)$ *with* $A \in \Delta_1 \subset \Delta$ *and* $\Delta_1 \cap D \neq \emptyset$. *If* Δ *is second erasable, then also* Δ_1 *is second erasable.*
2. *There exist elements* $E \in k[[x]]$ *and* $F \in M(k[[x']])$ *satisfying the following conditions. Let* $\hat{G} \in k[[x]]^L$ *be the element satisfying the relation 3.2 in Lemma 3.2, and* $\hat{P} = (x_1, x_2, \dots, x_{n-1}, x_n + F)$. *Obviously* $\hat{P}$ *is an ordered parameter system of* $k[[x]]$*:*

(a) *The Newton polyhedron* $\Gamma_+(\hat{G}, \hat{P})$ *is of Weierstrass type. Its characteristic vertex coincides with* A*.*
(b) *Every face* Δ *of* $\Gamma_+(\hat{G}, \hat{P})$ *is not second erasable.*
(c) *Every vertex of* $\Gamma_+(\hat{G}, \hat{P})$ *belonging to* D *is not first erasable.*
(d) *For every integer* j *with* $1 \leq j \leq L$ $\operatorname{coeff}(\hat{G}_j, \hat{P}, jA) = \operatorname{coeff}(G_j, P, jA)$.
(e) $\rho(\Gamma_+(\hat{G}, \hat{P}) \cap D) \subset \rho(\Gamma_+(G, P) \cap D)$.
(f) $\operatorname{ex}(E, \hat{P}) \subset D$. $\rho(\operatorname{ex}(E, \hat{P})) \subset \rho(\Gamma_+(G, P) \cap D)$.
(g) $\operatorname{ex}(F, P') \subset \rho(\Gamma_+(G, P) \cap D) - \rho(\Gamma_+(\hat{G}, \hat{P}) \cap D)$.

The unique closed point of an affine scheme $\operatorname{Spec}(k[[x]])$ is denoted by s_0. The completion of the local ring $\mathcal{O}_{\Sigma,\, s}$ of a scheme Σ at a point $s \in \Sigma$ is denoted by $\tilde{\mathcal{O}}_{\Sigma,\, s}$. Let $\sigma : \Sigma \to \operatorname{Spec}(k[[x]])$ be a morphism of schemes. A k-valued point $s \in \Sigma$ with $\sigma(s) = s_0$ is called a point *lying over* s_0. For any point $s \in \Sigma$ lying over s_0 we have a ring homomorphism $k[[x]] \to \tilde{\mathcal{O}}_{\Sigma,\, s}$ defined by pulling-back a function. We denote it by $\sigma^* : k[[x]] \to \tilde{\mathcal{O}}_{\Sigma,\, s}$.

Definition 3.8 Assume that an ordered parameter system $P = (x_1, x_2, \dots, x_n)$ of $k[[x]]$ is given.

1. Let i be an integer with $1 \leq i \leq n$. We say that an element $f \in k[[x]]$ is *simple at level* i with respect to P, if there is an element $g \in M(k[[x_1, x_2, \dots, x_{i-1}]])$ with $f = x_i + g$.
2. Let $I \subset \{1, 2, \dots, n\}$ be a subset with $\sharp I \geq 2$, and $C \subset \operatorname{Spec}(k[[x]])$ the closed smooth subscheme of codimension $\sharp I$ defined by the ideal in $k[[x]]$ generated by $\{x_i \mid i \in I\}$. Let $\sigma : \Sigma \to \operatorname{Spec}(k[[x]])$ denote the blowing-up of the scheme $\operatorname{Spec}(k[[x]])$ with center C. Let $s \in \Sigma$ be an arbitrary point lying over s_0. Let $\sigma^* : k[[x]] \to \tilde{\mathcal{O}}_{\Sigma,\, s}$ be the induced ring homomorphism. An ordered parameter system $\tilde{P} = (\xi_1, \xi_2, \dots, \xi_n)$ of $\tilde{\mathcal{O}}_{\Sigma,\, s}$ is said to be *induced* by P, if there is an integer $i \in I$ satisfying the following conditions:
    (a) $\sigma^* x_i = \xi_i$.
    (b) $\sigma^* x_j = \xi_j$, if $j \in \{1, 2, \dots, n\} - I$.
    (c) $\sigma^* x_j = \xi_i \xi_j$, if $j \in I$ and $j < i$.
    (d) $\sigma^* x_j = \xi_i(\xi_j + a_j)$ for some $a_j \in k$, if $j \in I$ and $j > i$.

Remark 1. In the above situation $\tilde{\mathcal{O}}_{\Sigma,\, s} \cong k[[x]]$ as topological k-algebras.
2. Depending on the subset $I \subset \{1, 2, \dots, n\}$ and the point $s \in \Sigma$, the induced ordered parameter system $\tilde{P}$ of $\tilde{\mathcal{O}}_{\Sigma,\, s}$ exists, and is uniquely determined by the ordered parameter system P of $k[[x]]$.

Let m be a positive integer. The *lexicographic order* on $\mathbb{Q}^m$ is a total order $\leq$ with the following properties: Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$, $\beta = (\beta_1, \beta_2, \dots, \beta_m) \in \mathbb{Q}^m$. $\alpha \leq \beta$, if and only if, $\alpha = \beta$ or $\alpha < \beta$. $\alpha < \beta$, if and only if, there is an integer i with $1 \leq i \leq m$ such that $\alpha_j = \beta_j$ for every j with $1 \leq j < i$ and $\alpha_i < \beta_i$. Expressions $\beta \geq \alpha$ and $\beta > \alpha$ are equivalent to $\alpha \leq \beta$ and $\alpha < \beta$ respectively. We can check that with this lexicographic order $\mathbb{Q}^m$ is a totally ordered abelian group.

Lemma 3.9 *Let* α_i $(i = 1, 2, 3, \dots)$ *be an infinite sequence of elements in* $\mathbb{Z}_0^m$*. Assume that* $\alpha_i \geq \alpha_{i+1}$ *for every* i*, where* $\geq$ *is the lexicographic order. Then, there exists a number* i_0 *such that* $\alpha_i = \alpha_{i_0}$ *for every number* i *with* $i \geq i_0$*.*

In the part below of this article, we assume that $n = 3$, $k[[x]] = k[[x_1, x_2, x_3]]$, and consider hypersurface singularities of dimension two.

Definition 3.10 1. A quadruple (g, g', h, P) satisfying the following conditions is called a *first standard decomposition*:
   (a) The last term $P = (x_1, x_2, x_3)$ is an ordered parameter system of the ring $k[[x]]$.
   (b) The first term g is an element of $k[[x]]$ such that there exist a non-negative integer ℓ, and ℓ elements $g_1, g_2, \dots, g_\ell \in M(k[[x_1, x_2]])$ satisfying $g = x_3^\ell + \sum_{j=1}^{\ell} g_j x_3^{\ell - j}$.
   (c) The second term g' is an element of $k[[x]]$, and is a product of a finite number of simple elements at level 3. In other words, there exist a non-negative integer m, and m elements $g'_1, g'_2, \dots, g'_m \in M(k[[x_1, x_2]])$ satisfying $g' = \prod_{j=1}^{m} (x_3 + g'_j)$
   (d) The third term h is an element of $k[[x]]$. There is an invertible element $u \in k[[x]]^*$ such that h/u is a product of a finite number of simple elements at level strictly less than 3.
2. Let $f \in k[[x]]$ be an element with $f \neq 0$. We say that a first standard decomposition (g, g', h, P) is one *of* f, if $f = gg'h$ holds.
3. If h has a normal crossing with respect to P, then we say that a first standard decomposition (g, g', h, P) is *of normal crossing type.*
4. The *multiplicity* $\mu(g, g', P)$ of a first standard decomposition (g, g', h, P) is an element (ℓ, m) in the set $\mathbb{Z}_0^2$ equipped with the lexicographic order, where ℓ and m are integers appearing in the above conditions.

The Gauss symbol $[r]$ of a real number $r \in \mathbb{R}$ is defined to be

$$[r] = \max\{\alpha \in \mathbb{Z} \mid \alpha \leq r\}.$$

Definition 3.11 1. A sextuple (g, g', h, G, H, P) satisfying the following conditions is called a *second standard decomposition*:
   (a) The quadruple (g, g', h, P) is a first standard decomposition. Let $\mu(g, g', P) = (\ell, m)$ and $L = \ell + m$.
   (b) The fifth term H is an element of $k[[x_1, x_2]]$. There is an invertible element $U \in k[[x_1, x_2]]^*$ such that H/U is a product of a finite number of simple elements at level strictly less than 3.
   (c) The fourth term $G = (G_1, G_2, \dots, G_L)$ is an element in $k[[x_1, x_2]]^L$. They satisfy

$$gg' = x_3^L + \sum_{j=1}^{L} H^j G_j x_3^{L-j},$$

   and $[\mathrm{ord}_{x_1}(G)] = [\mathrm{ord}_{x_2}(G)] = 0$.
2. Let $f \in k[[x]]$ be an element with $f \neq 0$. We say that a second standard decomposition (g, g', h, G, H, P) is one *of* f, if $f = gg'h$ holds.
3. If h and H have a normal crossing with respect to P, then we say that a second standard decomposition (g, g', h, G, H, P) is *of normal crossing type.*

Remark For any second standard decomposition (g, g', h, G, H, P) we have $G \neq 0$ and $H \neq 0$ by definition.

4 Resolution theorems

Under definitions in the previous section, the proof for the resolution of a germ of a two-dimensional hypersurface singularity over an arbitrary infinite field k

becomes a chain of propositions. By propositions below together with Lemma 3.9 we conclude that the resolution is always possible. Repeating blowing-up of the space and pulling-back the function finite times, any function is reduced to one with a normal crossing.

In this section we consider only the case $n = 3$. Let $k[[x]] = k[[x_1, x_2, x_3]]$. For an ordered parameter system $Q = (y_1, y_2, y_3)$ of $k[[x]]$, we write $Q' = (y_1, y_2)$. This Q' is an ordered parameter system of the subring $k[[y_1, y_2]]$.

Lemma 4.1 *Let* $f \in k[[x]]$ *be a non-zero element. Assume that there exist* $m + 1$ *elements* $f_0, f_1, \dots, f_m \in k[[x]]$ *such that* $f = f_0 f_1 \cdots f_m$, $\mathrm{ord}(f_i) = 1$ *for* $1 \le i \le m$, *and* $\ell = \mathrm{ord}(f_0)$. *Then, there exists a first standard decomposition* (g, g', h, P) *of* f *such that* $\mu(g, g', P) \le (\ell, m)$, *and* $h \in k[[x]]^*$.

Proof It follows from Lemma 2.8 and Corollary 2.7. □

Lemma 4.2 1. *Every non-zero element* $f \in k[[x]]$ *has a first standard decomposition of normal crossing type.*

2. *Assume that a non-zero element* $f \in k[[x]]$ *does not have a normal crossing. Let* (g, g', h, P) *be a first standard decomposition of normal crossing type of* f. *Then, there exist elements* G, H *and* $\hat{P}$ *such that* $(g, g', h, G, H, \hat{P})$ *is a second standard decomposition of normal crossing type of* f, $\hat{P}' = P'$, *and the Newton polyhedron* $\Gamma_+(G, \hat{P}')$ *has no first erasable face.*
3. *Let* (g, g', h, G, H, P) *be a second standard decomposition,* $P = (x_1, x_2, x_3)$, $\mu(g, g', P) = (\ell, m)$, *and* $L = \ell + m$. *Assume that the Newton polyhedron* $\Gamma_+(G, P')$ *has no first erasable face. Then, there are elements* $\hat{G} \in k[[x_1, x_2]]^L$, $\hat{H} \in k[[x_1, x_2]]$, $E \in M(k[[x_1, x_2]])$, *and* $F \in M(k[[x_1]])$ *satisfying the following conditions. Put* $\hat{P} = (x_1, x_2 + F, x_3 + E) = (\hat{x}_1, \hat{x}_2, \hat{x}_3)$:
   - (a) *Sextuple* $(g, g', h, \hat{G}, \hat{H}, \hat{P})$ *is a second standard decomposition.*
   - (b) $\mu(g, g', \hat{P}) = (\ell, m)$.
   - (c) $\mathrm{ord}_{\hat{x}_1}(\hat{G}) = \mathrm{ord}_{x_1}(G)$.
   - (d) $\mathrm{ord}_{\hat{x}_2}(\mathrm{in}_{\hat{x}_1}^{\hat{P}'}(\hat{G})) \le \mathrm{ord}_{x_2}(\mathrm{in}_{x_1}^{P'}(G))$.
   - (e) *The Newton polyhedron* $\Gamma_+(\hat{G}, \hat{P}')$ *has no first erasable face.*
   - (f) *The Newton polyhedron* $\Gamma_+(\hat{G}, \hat{P}')$ *has no second erasable face.*
4. *If* $\mathrm{ord}_{x_2}(\mathrm{in}_{x_1}^{P'}(G)) \ge 1$ *for a second standard decomposition* (g, g', h, G, H, P) *with* $P = (x_1, x_2, x_3)$, *then the Newton polyhedron* $\Gamma_+(G, P')$ *has two or more vertices.*

Proof 1. It follows from Lemma 4.1.
2. It follows from Theorem 3.3.2.
3. It folows from Theorem 3.7.2.
4. Easy. □

Remark Let (g, g', h, G, H, P) be a standard decomposition of f with $\mu(g, g', P) = (\ell, m)$, and $L = \ell + m$. The characteristic vertex of $\Gamma_+(f, P)$ is $(0, 0, L)$. The characteristic vertex of $\Gamma_+(G, P') = \rho(\Gamma_+(f, P) \cap D)$ is $(\mathrm{ord}_{x_1}(G), \mathrm{ord}_{x_2}(\mathrm{in}_{x_1}^{P'}(G)))$.

Theorem 4.3 *Let* (g, g', h, G, H, P) *be a second standard decomposition of normal crossing type. Assume that the Newton polyhedron* $\Gamma_+(G, P')$ *has no first erasable face, and* $\mathrm{ord}(G) < 1$. *We put* $f = gg'h$.

1. *If, moreover,* $\mathrm{ord}_{x_1}(H) = \mathrm{ord}_{x_2}(H) = 0$*, then there exists a first standard decomposition* $(\hat{g}, \hat{g}', \hat{h}, \hat{P})$ *of normal crossing type of* f *such that* $\mu(\hat{g}, \hat{g}', \hat{P}) < \mu(g, g', P)$*.*
2. *Assume that* $i = 1$ *or* 2*, and* $\mathrm{ord}_{x_i}(H) > 0$*. Let* $\sigma : \Sigma \to \mathrm{Spec}(k[[x]])$ *be the blowing-up of the scheme* $\mathrm{Spec}(k[[x]])$ *with center in the closed smooth subscheme defined by* $x_i = x_3 = 0$*. For every point* $s \in \Sigma$ *lying over* s_0 *one of the following claims holds:*
   - (a) *The element* $\sigma^* f \in \tilde{\mathcal{O}}_{\Sigma,\, s}$ *has a normal crossing.*
   - (b) *There exists a first standard decomposition* $(\tilde{g}, \tilde{g}', \tilde{h}, \tilde{P})$ *of normal crossing type of* $\sigma^* f$ *at the ring* $\tilde{\mathcal{O}}_{\Sigma,\, s}$ *such that* $\mu(\tilde{g}, \tilde{g}', \tilde{P}) < \mu(g, g', P)$*.*
   - (c) *There exists a second standard decomposition* $(\tilde{g}, \tilde{g}', \tilde{h}, \tilde{G}, \tilde{H}, \tilde{P})$ *of* $\sigma^* f$ *at the ring* $\tilde{\mathcal{O}}_{\Sigma,\, s}$ *satisfying the following conditions. We write* $\tilde{P} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$*:*
     - (i) *It is of normal crossing type.*
     - (ii) *The Newton polyhedron* $\Gamma_+(\tilde{G}, \tilde{P}')$ *has no first erasable face.*
     - (iii) $\mathrm{ord}(\tilde{G}) = \mathrm{ord}(G) < 1$*.*
     - (iv) $\mu(\tilde{g}, \tilde{g}', \tilde{P}) \leq \mu(g, g', P)$*.*
     - (v) $\mathrm{ord}_{\tilde{x}_i}(\tilde{H}) = \mathrm{ord}_{x_i}(H) - 1$*.*
     - (vi) $\mathrm{ord}_{\tilde{x}_j}(\tilde{H}) = \mathrm{ord}_{x_j}(H)$*, where* j *is the integer satisfying* $\{1, 2\} = \{i, j\}$*.*

Proof Claim 1 follows from Lemma 4.1. □

Theorem 4.4 *Let* (g, g', h, G, H, P) *be a second standard decomposition of normal crossing type,* $f = gg'h$ *and* $P = (x_1, x_2, x_3)$*. Assume that the Newton polyhedron* $\Gamma_+(G, P')$ *has no first erasable face,* $\mathrm{ord}(G) \geq 1$*, and* $\mathrm{ord}_{x_2}(\mathrm{in}_{x_1}^{P'}(G)) < 1$*.*

Let $\sigma : \Sigma \to \mathrm{Spec}(k[[x]])$ *be the blowing-up of the scheme* $\mathrm{Spec}(k[[x]])$ *with center in the closed smooth subscheme defined by* $x_1 = x_2 = 0$*. For every point* $s \in \Sigma$ *lying over* s_0 *there exists a second standard decomposition* $(\tilde{g}, \tilde{g}', \tilde{h}, \tilde{G}, \tilde{H}, \tilde{P})$ *of* $\sigma^* f$ *at the ring* $\tilde{\mathcal{O}}_{\Sigma,\, s}$ *satisfying the following conditions:*

1. *It is of normal crossing type.*
2. *The Newton polyhedron* $\Gamma_+(\tilde{G}, \tilde{P}')$ *has no first erasable face.*
3. $\mu(\tilde{g}, \tilde{g}', \tilde{P}) \leq \mu(g, g', P)$*.*
4. *Let* $\tilde{P} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$*.*

   $$(\mathrm{ord}_{\tilde{x}_2}(\mathrm{in}_{\tilde{x}_1}^{\tilde{P}'}(\tilde{G})), \mathrm{ord}_{\tilde{x}_1}(\tilde{G})) < (\mathrm{ord}_{x_2}(\mathrm{in}_{x_1}^{P'}(G)), \mathrm{ord}_{x_1}(G)),$$

   where $<$ *is the lexicographic order.*

Let (g, g', h, G, H, P) be a second standard decomposition, and $P = (x_1, x_2, x_3)$.

By definition there is a unique set of an element $u \in k[[x]]^*$, non-negative integers α_1, α_2, and α_2 elements $h_1, h_2, \ldots, h_{\alpha_2} \in M(k[[x_1]])$ satisfying

$$hH = u x_1^{\alpha_1} \prod_{i=1}^{\alpha_2} (x_2 + h_i).$$

1. We write $\delta(h, H, P) = \sharp\,\{h_i \mid 1 \leq i \leq \alpha_2\} \in \mathbb{Z}_0$.
2. We say that (g, g', h, G, H, P) is *of regular type*, if, either $\alpha_2 = 0$, or, $\alpha_2 > 0$ and $h_i = 0$ for some i with $1 \leq i \leq \alpha_2$.
3. We define $\epsilon(h, H, P) = \min\{\mathrm{ord}(h_i) \mid 1 \leq i \leq \alpha_2\}$ if $\alpha_2 > 0$, and $\epsilon(h, H, P) = \infty$ if $\alpha_2 = 0$.

Lemma 4.5 1. *The following three conditions are equivalent:*

(a) *The second standard decomposition* (g, g', h, G, H, P) *is of normal crossing type.*

(b) $\delta(h, H, P) \leq 1$ *and* (g, g', h, G, H, P) *is of regular type.*

(c) $\epsilon(h, H, P) = \infty$.

2. *Assume that the Newton polyhedron* $\Gamma_+(G, P')$ *has no first erasable face, and* $\mathrm{ord}_{x_2}(\mathrm{in}_{x_1}^{P'}(G)) < 1$.

There exists a second standard decomposition $(\hat{g}, \hat{g}', \hat{h}, \hat{G}, \hat{H}, \hat{P})$ *of* $gg'h$ *satisfying the following conditions. We write* $\hat{P} = (\hat{x}_1, \hat{x}_2, \hat{x}_3)$*:*

(a) $\mu(\hat{g}, \hat{g}', \hat{P}) \leq \mu(g, g', P)$

(b) *The Newton polyhedron* $\Gamma_+(\hat{G}, \hat{P}')$ *has no first erasable face.*

(c) $\mathrm{ord}_{\hat{x}_2}(\mathrm{in}_{\hat{x}_1}^{\hat{P}'}(\hat{G})) = \mathrm{ord}_{x_2}(\mathrm{in}_{x_1}^{P'}(G)) < 1$

(d) $\delta(\hat{h}, \hat{H}, \hat{P}) = \delta(h, H, P)$

(e) *The second standard decomposition* $(\hat{g}, \hat{g}', \hat{h}, \hat{G}, \hat{H}, \hat{P})$ *is of regular type.*

Theorem 4.6 *Let* (g, g', h, G, H, P) *be a second standard decomposition, and* $P = (x_1, x_2, x_3)$. *Assume that the Newton polyhedron* $\Gamma_+(G, P')$ *has no first erasable face,* $\mathrm{ord}_{x_2}(\mathrm{in}_{x_1}^{P'}(G)) < 1$, (g, g', h, G, H, P) *is not of normal crossing type, and it is of regular type. We put* $f = gg'h$.

Let $\sigma : \Sigma \to \mathrm{Spec}(k[[x]])$ *be the blowing-up of the scheme* $\mathrm{Spec}(k[[x]])$ *with center in the closed smooth subscheme defined by* $x_1 = x_2 = 0$. *For every point* $s \in \Sigma$ *lying over* s_0 *there exists a second standard decomposition* $(\tilde{g}, \tilde{g}', \tilde{h}, \tilde{G}, \tilde{H}, \tilde{P})$ *of* $\sigma^* f$ *at the ring* $\tilde{\mathcal{O}}_{\Sigma, s}$ *satisfying the following conditions. We write* $\tilde{P} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$*:*

1. *The Newton polyhedron* $\Gamma_+(\tilde{G}, \tilde{P}')$ *has no first erasable face.*
2. $\mathrm{ord}_{\tilde{x}_2}(\mathrm{in}_{\tilde{x}_1}^{\tilde{P}'}(\tilde{G})) < 1$.
3. $\mu(\tilde{g}, \tilde{g}', \tilde{P}) \leq \mu(g, g', P)$.
4. *One of the following three conditions holds:*
   (a) $(\tilde{g}, \tilde{g}', \tilde{h}, \tilde{G}, \tilde{H}, \tilde{P})$ *is of normal crossing type.*
   (b) $\delta(\tilde{h}, \tilde{H}, \tilde{P}) < \delta(h, H, P)$.
   (c) *The following conditions hold:*
      (i) $(\tilde{g}, \tilde{g}', \tilde{h}, \tilde{G}, \tilde{H}, \tilde{P})$ *is not of normal crossing type.*
      (ii) *It is of regular type.*
      (iii) $\delta(\tilde{h}, \tilde{H}, \tilde{P}) = \delta(h, H, P)$
      (iv) $\epsilon(\tilde{h}, \tilde{H}, \tilde{P}) = \epsilon(h, H, P) - 1$

Let (g, g', h, G, H, P) be a second standard decomposition with $\mathrm{ord}_{x_2}(\mathrm{in}_{x_1}^{P'}(G)) \geq 1$. The Newton polyhedron $\Gamma_+(G, P')$ has two or more vertices. Let $A_i = (A_{1,i}, A_{2,i}) \in \mathbb{R}_0^2$ $(i = 1, 2, \ldots, s)$ be its vertices, where s is the number of vertices. We can assume that

$$\begin{array}{ccccccc} A_{1,1} & < & A_{1,2} & < & \cdots & < & A_{1,s}, \\ A_{2,1} & > & A_{2,2} & > & \cdots & > & A_{2,s}. \end{array}$$

We write

$$e(G, P') = -\frac{A_{1,1} - A_{1,2}}{A_{2,1} - A_{2,2}}.$$

Remark The vertex $A_1 = (\mathrm{ord}_{x_1}(G), \mathrm{ord}_{x_2}(\mathrm{in}_{x_1}^{P'}(G)))$ is the characteristic vertex of $\Gamma_+(G, P')$. The characteristic vertex of $\rho(\Gamma_+(G, P') \cap D)$ can be identified with $e(G, P')$.

Theorem 4.7 *Let* (g, g', h, G, H, P) *be a second standard decomposition, and* $P = (x_1, x_2, x_3)$. *Assume that the Newton polyhedron* $\Gamma_+(G, P')$ *has no first erasable face, it has no second erasable face, and* $\mathrm{ord}_{x_2}(\mathrm{in}_{x_1}^{P'}(G)) \geq 1$. *We write* $f = gg'h$.

Let $\sigma : \Sigma \to \mathrm{Spec}(k[[x]])$ *be the blowing-up of the scheme* $\mathrm{Spec}(k[[x]])$ *with center in the closed smooth subscheme defined by* $x_1 = x_2 = 0$. *For every point* $s \in \Sigma$ *lying over* s_0 *there exists a second standard decomposition* $(\tilde{g}, \tilde{g}', \tilde{h}, \tilde{G}, \tilde{H}, \tilde{P})$ *of* $\sigma^* f$ *at the ring* $\tilde{\mathcal{O}}_{\Sigma,\, s}$ *satisfying the following conditions. We write* $\tilde{P} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$*:*

1. *The Newton polyhedron* $\Gamma_+(\tilde{G}, \tilde{P}')$ *has no first erasable face.*
2. $\mu(\tilde{g}, \tilde{g}', \tilde{P}) \leq \mu(g, g', P)$.
3. *It satisfies one of the following two conditions:*
    (a) $\mathrm{ord}_{\tilde{x}_2}(\mathrm{in}_{\tilde{x}_1}^{\tilde{P}'}(\tilde{G})) < \mathrm{ord}_{x_2}(\mathrm{in}_{x_1}^{P'}(G))$.
    (b) *The following conditions holds:*
        (i) $\mathrm{ord}_{\tilde{x}_2}(\mathrm{in}_{\tilde{x}_1}^{\tilde{P}'}(\tilde{G})) = \mathrm{ord}_{x_2}(\mathrm{in}_{x_1}^{P'}(G))$.
        (ii) *The Newton polyhedron* $\Gamma_+(\tilde{G}, \tilde{P}')$ *has no second erasable face.*
        (iii) $e(\tilde{G}, \tilde{P}') = e(G, P') - 1$.

In the proof of above theorems the concept of induced ordered parameter systems in Definition 3.8.2 is effective.

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