HYPERSURFACE SINGULARITIES WHICH CANNOT BE RESOLVED IN CHARACTERISTIC POSITIVE

TOHSUKE URABE

Dedicated to Professor Takuo Fukuda on his sixtieth birthday

Abstract. Assume that there exists a hypersurface singularity which cannot be resolved by iterated monoidal transformations in positive characteristic. We show that in the set of defining functions of hypersurface singularities which cannot be resolved, we can find a function satisfying very strong conditions. By these conditions we may be able to deduce a contradiction under the above assumption. Besides, we introduce essential concepts for the study of resolution of singularities of germs such as Weierstrass representations, the Abhyankar condition, "not reductive" and so forth.

1. Introduction

In this paper we consider the problem of resolution of singularities in the case of positive characteristic. Since Hironaka's success in resolution of singularities in characteristic zero (Hironaka [5].), the solution of this problem has long been expected. However, it is still open.

This is perhaps because the problem is hard by nature.

Therefore, we consider the problem in the category of germs of varieties. Besides, we assume that the singularity is a hypersurface.

Recently very clear exposition was given in Appendix of Abhyankar [2] for resolution of hypersurface singularities in characteristic zero. I have noticed that most of Abhyankar's ideas are effective even in characteristic positive. I supply the missing part to his ideas, and develop the theory of positive characteristic in this paper.

Throughout this paper we fix the ground field k. We assume that k is an algebraically closed field and the characteristic number p of k is positive. The prime number p is used to denote the characteristic number of k.

By B we denote a ring isomorphic to the formal power series ring $k[[x_0, x_1, \dots, x_r]]$ over k with $r+1$ variables $x_0, x_1, \dots, x_r$ for some non-negative integer r. Note that $r+1 = \dim B$. The affine scheme $\mathrm{Spec}(B)$ associated with B is the space whose ring of functions coincides with B.

Assume that a ring B and a non-zero element $f \in B$ are given. We consider the following game, which we will call a *resolution game* of $f \in B$. It has two players $\langle I \rangle$ and $\langle II \rangle$, and they repeat steps under the same rule. At each step, if f has a normal crossing in B, then the player $\langle I \rangle$ wins and they terminate the game. Otherwise, $\langle I \rangle$ chooses a non-singular closed subscheme of $\mathrm{Spec}(B)$. Let $\sigma : \Sigma \to \mathrm{Spec}(B)$ be

---

1991 *Mathematics Subject Classification.* Primary 14B05; Secondary 32S05, 13H99.

Key words and phrases. hypersurface, singularity, resolution of singularities, monoidal transformation, normal crossing, Weierstrass preparation theorem, positive characteristic.

the blowing-up of the scheme Spec(B) with center in the chosen subscheme. Then, the player $\langle II\rangle$ chooses a closed point $s \in \Sigma$ lying over the unique closed point of Spec(B). After replacing the pair (B, f) by the pair (B', f') where B' denotes the completion of the local ring $\mathcal{O}_{\Sigma,s}$ of Σ at s and f' denotes the image of f by the ring homomorphism $B \to \mathcal{O}_{\Sigma,s} \to B'$, they proceed to the next step.

If $\langle I\rangle$ can *always* win the game whatever closed points $\langle II\rangle$ chooses, then we say that $f \in B$ *can be resolved* by iterated monoidal transformations, or we say f can be resolved, in short. Note that we have decided to work in the category of germs, and we have to use the concept of resolution games to formulate the words "f can be resolved".

A ring homomorphism $B \to B'$ in the above form is called a *monoidal transformation.* Note that it is defined depending on a non-singular closed subscheme of Spec(B) and a closed point $s \in \Sigma$. A composition of a finite number of monoidal transformations is called an *iterated monoidal transformation.*

By A^* we denote the set of invertible elements of a ring A.

Theorem 1.1. *Assume that the set $\mathcal{X}$ of pairs (B, f) such that $f \neq 0$ and $f \in B$ cannot be resolved is* not *empty. Let $n+1 = \min\{\dim B \mid (B, f) \in \mathcal{X}\}$. There exists an element $(B, g) \in \mathcal{X}$ with $n+1 = \dim B$ such that there are an element $t \in B$, a positive integer m, an infinite sequence $\{B_\nu \mid 0 \leq \nu < \infty\}$ of complete regular local k-algebras with $B_0 = B$, and an infinite sequence $\{B_\nu \to B_{\nu+1} \mid 0 \leq \nu < \infty\}$ of iterated monoidal transformations with the following properties: We denote an element in B_ν and its image in $\mathcal{B} = \varinjlim_\nu B_\nu = \cup_{\nu=0}^\infty B_\nu$ or in B_μ for $\mu \geq \nu$ by the same symbol. There exist an infinite sequence*

$$\{P_\nu = \{x_{\nu,1}, x_{\nu,2}, \ldots, x_{\nu,n}\} \mid 0 \leq \nu < \infty\}$$

of finite subsets $P_\nu \subset B_\nu$, an infinite sequence

$$\{H_\nu \mid 1 \leq \nu < \infty\}$$

of elements $H_\nu \in B_\nu$, an infinite sequence

$$\{(b_{\nu,1}, b_{\nu,2}, \ldots, b_{\nu,n}) \mid 1 \leq \nu < \infty\}$$

of n-tuples $(b_{\nu,1}, b_{\nu,2}, \ldots, b_{\nu,n})$ of non-negative integers $b_{\nu,i}$, an infinite sequence

$$\{\eta_\nu \mid 1 \leq \nu < \infty\}$$

of elements $\eta_\nu \in B_\nu$, and an infinite sequence

$$\{(F_{\nu,1}, F_{\nu,2}, \ldots, F_{\nu,m}) \mid 0 \leq \nu < \infty\}$$

of m-tuples $(F_{\nu,1}, F_{\nu,2}, \ldots, F_{\nu,m})$ of elements $F_{\nu,j} \in B_\nu$ satisfying the conditions (1.ν), (5.ν) *below for every non-negative integer ν, and the conditions* (2.ν), (3.ν), (4.ν), (6.ν) *below for every positive integer ν.*

(1.ν) *The set $P_\nu \cup \{t\} = \{x_{\nu,1}, x_{\nu,2}, \ldots, x_{\nu,n}, t\}$ is a parameter system of a regular local ring B_ν. For simplicity, we write $Z_\nu = k[[x_{\nu,1}, x_{\nu,2}, \ldots, x_{\nu,n}]]$ (the closed k-subalgebra of B_ν generated by P_ν), and*

$$x_\nu^{b_\nu} = x_{\nu,1}^{b_{\nu,1}} x_{\nu,2}^{b_{\nu,2}} \cdots x_{\nu,n}^{b_{\nu,n}} = \prod_{i=1}^{n} x_{\nu,i}^{b_{\nu,i}}.$$

(2.ν) *For every $1 \leq i \leq n$, $x_{\nu-1,i} \in Z_\nu$ and $x_{\nu-1,i}$ has a normal crossing with respect to the parameter system P_ν of Z_ν.*

(3.ν) *At least one of* $b_{\nu,1}, b_{\nu,2}, \dots, b_{\nu,n}$ *is not zero.* (*Note that* $x_\nu^{b_\nu}$ *belongs to the maximal ideal of* Z_ν.)

(4.ν)

$$H_\nu = \eta_\nu \prod_{\mu=1}^{\nu} x_\mu^{b_\mu}, \quad \eta_\nu \in Z_\nu^*.$$

(*Note that* H_ν *belongs to the* ν*-th power of the maximal ideal of* Z_ν.)

(5.ν) *We write* $t_\nu = t + \sum_{\mu=1}^{\nu} H_\mu \in Z_\nu$. *The pair* (g, t_ν) *is a separable Weierstrass pair over* (B_ν, P_ν). *The element* $F_{\nu,j}$ *belongs to the maximal ideal of* Z_ν *for every* $1 \leq j \leq m$, *and*

$$g = t_\nu^m + \sum_{j=1}^{m} \left(\prod_{\mu=1}^{\nu} x_\mu^{b_\mu} \right)^j F_{\nu,j} t_\nu^{m-j}.$$

We understand $\sum_{\mu=1}^{0} H_\mu = 0$ *and* $\prod_{\mu=1}^{0} x_\mu^{b_\mu} = 1$ *here as a convention.*

(6.ν) (*Note that* $F_{\nu-1,j} \in Z_{\nu-1} \subset Z_\nu$ *for every* $1 \leq j \leq m$.) *For every* $1 \leq j \leq m$ $\frac{F_{\nu-1,j}}{(x_\nu^{b_\nu})^j} \in Z_\nu$, *and for some* $1 \leq j \leq m$ $\frac{F_{\nu-1,j}}{(x_\nu^{b_\nu})^j} \in Z_\nu^*$.

Remark . 1. See Definition 2.16 and Definition 2.19 for the terminology "separable Weierstrass pair" in (5.ν).

2. By the equality in (5.ν) it is intuitively quite plausible that

$$g = \lim_{\nu \to \infty} \left\{ t_\nu^m + \sum_{j=1}^{m} \left(\prod_{\mu=1}^{\nu} x_\mu^{b_\mu} \right)^j F_{\nu,j} t_\nu^{m-j} \right\} = (\lim_{\nu \to \infty} t_\nu)^m = (t + \sum_{\nu=1}^{\infty} H_\nu)^m,$$

since $\mathrm{ord}_{Z_\nu}(\left(\prod_{\mu=1}^{\nu} x_\mu^{b_\mu}\right)^j F_{\nu,j}) \geq j\nu$, $\mathrm{ord}_{Z_\nu}(H_\nu) \geq \nu$, and $\mathrm{ord}_{Z_\nu} \geq \mathrm{ord}_{Z_\mu}$ for $\nu \geq \mu$.

If this is true, then we can conclude that $g = (t + H)^m$ for some $H \in Z_0$ by a refinement of the Weierstrass preparation theorem in positive characteristic (Lemma 2.20), and that the set $\mathcal{X}$ is *empty*, since $g = (t + H)^m \in B$ has a normal crossing and thus g can be resolved by definition.

However, it is very hard to give a meaning to the above limit, since the ring $\mathcal{Z} = \varinjlim_\nu Z_\nu$ is not necessarily noetherian.

We could improve the above result, if we could find wiser strategy for the player $\langle I \rangle$. However, I believe that the latter remark above is the essential problem of the resolution theory, and I write this paper to focus targets of the future study and to establish essential concepts such as the Abhyankar condition and "not reductive". In the subsequent paper we will try to overcome this difficulty, and will give some results.

The rings B_ν's are complete, because we apply the Weierstrass preparation theorem in the essential part of our theory.

I hope that this paper together with several related results in a few years (Kuhlmann [6], Spivakovsky [8].) will play a key role to open the door toward the final solution of resolution of singularities.

Let Γ be a totally ordered abelian group. Often in this paper we associate a special element ∞ called the *infinity* with Γ. We assume that the element ∞ satisfies the following two conditions:

1. $\infty \geq \infty$, $\infty + \infty = \infty$, and $n\infty = \infty n = \infty$ for every positive integer n.
2. For every element $\gamma \in \Gamma$ $\gamma \neq \infty$, $\gamma < \infty$ and $\gamma + \infty = \infty + \gamma = \infty$.

Any positive integer m can be written uniquely in the form $m = lq$ where l is an integer prime to p and $q = p^\delta$ is a power of p. The exponent δ is a non-negative integer. The pair (l, q) is called the *p-decomposition* of m. For simplicity we say that $m = lq$ is the p-decomposition of m, or that $m = lp^\delta$ is the p-decomposition of m.

By $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ we denote the ring of integers, the rational number field and the real number field respectively. By $\mathbb{Z}_0$ we denote the set of non-negative integers.

The number of elements in a finite set X is denoted by $\sharp X$.

Section 1 is the introduction. Sections 2 is a preliminary part. We collect results on commutative rings. In Section 3 we give definitions for fundamental objects in our formulation such as a space germ, a framed space germ, a basic space germ, an elementary monoidal transform, an iterated elementary monoidal transform (an IEMT, in short), an iterated analytic monoidal transform (an IAMT, in short), and so forth. In Section 4 we explain the concept of framed resolution games, which plays an important role in this paper. The concept of Weierstrass representations is introduced in Section 5. This concept plays the central role in our theory. The Abhyankar condition, the strong Abhyankar condition and the condition "reductive" are defined for Weierstrass representations. The Abhyankar condition is essentially equivalent to the claim $(6.\nu)$ in Theorem 1.1. The condition "not reductive" is very important in characteristic positive. Let $G_1, G_2, \ldots, G_m \in Z = k[[x_1, x_2, \ldots, x_n]]$ and $g = t^m + \sum_{j=1}^m G_j t^{m-j}$. We define $\mathrm{w}(g,\, t) = \min\{\mathrm{ord}(G_j)/j \mid 1 \leq j \leq m\} \in \mathbb{Q} \cup \{\infty\}$. Note that for any $H \in Z$ there are elements $G'_1, G'_2, \ldots, G'_m \in Z$ with $g = (t + H)^m + \sum_{j=1}^m G'_j (t + H)^{m-j}$. Thus $\mathrm{w}(g,\, t + H) = \min\{\mathrm{ord}(G'_j)/j \mid 1 \leq j \leq m\}$ is defined. The pair (g, t) is said to be *reductive*, if there is an element $H \in Z$ with $\mathrm{w}(g,\, t) < \mathrm{w}(g,\, t + H)$. In Section 6 we show that if a Weierstrass representation satisfies both the Abhyankar condition and the condition "not reductive", then the singularity can be resolved even in characteristic positive. By this result our goal becomes to modify a given Weierstrass representation to satisfy both conditions. An effective tool called a reduction sequence is introduced in Section 7 to treat both the Abhyankar condition and the condition "not reductive" simultaneously. Central ideas of a reduction sequence are like the following: Let $Z_0 = Z$. We consider a pair $(g, t_0) = (g, t)$ as above. Assume that we have already obtained rings $Z_0, Z_1, \ldots, Z_{\nu-1}$ and pairs $(g, t_0), (g, t_1), \ldots, (g, t_{\nu-1})$. By the induction hypothesis on dimensions there exists an IAMT Z_ν of $Z_{\nu-1}$ such that the pair $(g, t_{\nu-1})$ satisfies the Abhyankar condition, if we regard the coefficient functions $G_{\nu-1,1}, G_{\nu-1,2}, \ldots, G_{\nu-1,m} \in Z_{\nu-1}$ of g as elements in Z_ν by the iterated monoidal transformation $Z_{\nu-1} \to Z_\nu$. If $(g, t_{\nu-1})$ is not reductive when we regard $G_{\nu-1,j}$'s as elements in Z_ν, then we terminate our procedure. Otherwise, we choose an element $H_\nu \in Z_\nu$ with $\mathrm{w}(g, t_{\nu-1}) < \mathrm{w}(g, t_{\nu-1} + H_\nu)$ and put $(g, t_\nu) = (g, t_{\nu-1} + H_\nu)$. The sequence obtained by repeating this procedure is called a *reduction sequence*. We will see in Section 8 that if there is a function which cannot be resolved, then there exists a reduction sequence with infinite length and that any member $G_{\nu,j}$ of the coefficient functions of (g, t_ν) has a very special form as in Theorem 1.1.

After I sent the first draft of this paper to the AMS preprint server, Joseph Lipman kindly informed me that Section 4 of Bierstone and Milman [3] (and Section 3 of Bierstone and Milman [4]) contains the same results as ones in Appendix of Abhyankar [2], and I received envelope containing a lot of results by Bierstone and

Milman from Bierstone. By Lemma 4.7 in Bierstone and Milman [3] (or Lemma 3.12 in Bierstone and Milman [4]) I could improve some descriptions in Section 3 and Section 5 of my manuscript. Because of difference on how to choose centers of monoidal transformations among in [2], in [3] and in [4], I take Abhyankar [2] as the basis of our theory in this paper. (The methods in [2] and [3] are simpler, but they choose centers even outside the zero-locus of the function under consideration. Two methods are slightly different. In the third method in [4] they choose centers inside the singular sets of the zero-locus of the function. The third can be regarded as the improvement of the second one in [3].)

This work was accomplished during my stay at Max-Planck-Institut für Mathematik at Bonn in the period of October 1998 to March 1999. I express thanks to Max-Planck-Institut for warm hospitality. I thank also Herwig Hauser for hospitality and interesting discussions we held at Innsbruck University.

2. Ring theory

In this section we review several important points in the ring theory and explain non-standard concepts.

By a ring we mean a commutative ring with the identity. We use standard terminologies in the commutative ring theory. (Matsumura [7]. Zariski-Samuel [10].)

Let A be a ring. By A^* we denote the set of invertible elements of A, i.e.,

$$A^* = \{a \in A \mid \text{ There exists an element } b \in A \text{ with } ab = 1\}.$$

A ring with the unique maximal ideal is called a *local ring*. For a local ring A by $M(A)$ we denote the maximal ideal of A. A local ring A is the disjoint union of A^* and $M(A)$.

By $\mathbb{Z}_0$ we denote the set of non-negative integers.

Let A be a ring. The binomial coefficient $\binom{r}{j}$ is the mapping $(r, j) \in \mathbb{Z}_0^2 \mapsto \binom{r}{j} \in A$ satisfying the following two conditions:

1. If $0 \leq r < j$, then $\binom{r}{j} = 0$.
2. For every non-negative integer r the equality
$$(t+1)^r = \sum_{j=0}^{r} \binom{r}{j} t^j$$
holds in the ring $A[t]$, where t is an algebraically independent element over A.

Lemma 2.1. *If* $0 \leq a \leq b$ *and* $a \leq r$, *then*

$$\binom{b}{a}\binom{r}{b} = \binom{r}{a}\binom{r-a}{b-a}.$$

Proof. Let t, u be algebraically independent elements over A. Computing $(tu+u+1)^r$ in two different ways we obtain the claim. □

Lemma 2.2. *Assume that a ring A contains a field of characteristic p. Let δ and l be non-negative integers. If a non-negative integer j is not a multiple of p^δ, then*

$$\binom{lp^\delta}{j} = 0.$$

If there is a non-negative integer i with $j = ip^\delta$, then

$$\binom{lp^\delta}{j} = \binom{lp^\delta}{ip^\delta} = \binom{l}{i}.$$

Let n be a non-negative integer, and $x_0, x_1, \dots, x_n$ $n+1$ of variables. We consider the formal power series ring $k[[x]] = k[[x_0, x_1, \dots, x_n]]$ over k. Any element $f \in k[[x]]$ can be written uniquely in the form

$$f = \sum_{\alpha \in \mathbb{Z}_0^{n+1}} f_\alpha x^\alpha,$$

where $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_n) \in \mathbb{Z}_0^{n+1}$, $\alpha_0, \alpha_1, \dots, \alpha_n$ are non-negative integers, $f_\alpha \in k$ for any $\alpha \in \mathbb{Z}_0^{n+1}$, and $x^\alpha = x_0^{\alpha_0} x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.

Lemma 2.3.
1. *The ring $k[[x]]$ is a noetherian complete regular local ring of dimension $n+1$, and is a unique factorization domain (UFD, in short.) with the unique maximal ideal $M(k[[x]]) = \{f \in k[[x]] \mid f_{(0,0,\dots,0)} = 0\}$. The set $\{x_0, x_1, \dots, x_n\}$ is a parameter system of $k[[x]]$.*
2. *Let $y_0, y_1, \dots, y_n$ be $n+1$ elements in $k[[x]]$. The following three conditions are equivalent:*
   - (a) *The set $\{y_0, y_1, \dots, y_n\}$ is a parameter system of $k[[x]]$.*
   - (b) *There exists a unique continuous isomorphism $\varphi : k[[x]] \to k[[x]]$ of k-algebras satisfying $\varphi(x_i) = y_i$ for every $0 \le i \le n$.*
   - (c) *$y_0, y_1, \dots, y_n \in M(k[[x]])$ and $\det(\partial y_i/\partial x_j) \notin M(k[[x]])$.*

Definition 2.4.
1. Let $Q = \{x_0, x_1, \dots, x_n\}$ be a parameter system of the ring $k[[x]]$ and $f \in k[[x]]$. We say that f has a *normal crossing with respect to Q*, if there are non-negative integers $\beta_0, \beta_1, \dots, \beta_n$ with

$$\frac{f}{x_0^{\beta_0} x_1^{\beta_1} \cdots x_n^{\beta_n}} \in k[[x]]^*.$$

2. An element $f \in k[[x]]$ is said to have a *normal crossing*, if there is a parameter system Q of $k[[x]]$ such that f has a normal crossing with respect to Q.

Lemma 2.5. *Let Q be a parameter system of $k[[x]]$, and $f \in k[[x]]$, $g \in k[[x]]$.*
1. *The following two conditions are equivalent:*
   - (a) *The product fg has a normal crossing with respect to Q.*
   - (b) *Both f and g have a normal crossing with respect to Q.*
2. *If the product fg has a normal crossing, then both f and g have a normal crossing.*

Proof. It follows from the uniqueness of the irreducible decomposition. □

Let $w = (w_0, w_1, \dots, w_n) \in \mathbb{Z}_0^{n+1}$, and $Q = \{x_0, x_1, \dots, x_n\}$ be a parameter system of $k[[x]]$. We will define a map

$$\mathrm{ord}_w : k[[x]] \to \mathbb{Z}_0 \cup \{\infty\}$$

associated with w and Q. The value $\operatorname{ord}_w(f) \in \mathbb{Z}_0 \cup \{\infty\}$ for $f \in k[[x]]$ is called the *order* of f with *weight* w. Let $f = \sum_{\alpha \in \mathbb{Z}_0^{n+1}} f_\alpha x^\alpha \in k[[x]]$ be an element. If $f = 0$, then we put $\operatorname{ord}_w(f) = \infty$. Assume $f \neq 0$. Let

$$\operatorname{ex}(f) = \{\alpha \in \mathbb{Z}_0^{n+1} \mid f_\alpha \neq 0\}.$$

Since $f \neq 0$, the set $\operatorname{ex}(f)$ is not empty. We define

$$\operatorname{ord}_w(f) = \min\{\sum_{i=0}^{n} w_i \alpha_i \mid (\alpha_0, \alpha_1, \dots, \alpha_n) \in \operatorname{ex}(f)\} \in \mathbb{Z}_0.$$

Lemma 2.6. 1. *If $f, g \in k[[x]]$, then $\operatorname{ord}_w(fg) = \operatorname{ord}_w(f) + \operatorname{ord}_w(g)$.*
2. *If $f, g \in k[[x]]$, then $\operatorname{ord}_w(f + g) \geq \min\{\operatorname{ord}_w(f), \operatorname{ord}_w(g)\}$.*
3. *Assume that $w_i = 0$ for $0 \leq i \leq m$ and $w_i > 0$ for $m + 1 \leq i \leq n$. We have $\operatorname{ord}_w(f) = 0$ if and only if $f_{(\alpha_0, \alpha_1, \dots, \alpha_m, 0, 0, \dots, 0)} \neq 0$ for some $m + 1$ non-negative integers $\alpha_0, \alpha_1, \dots, \alpha_m$.*
4. *Assume that $w_i >$ for every $0 \leq i \leq n$. We have $\operatorname{ord}_w(f) = 0$ if and only if $f \in k[[x]]^*$.*

Let E denote the field of fractions of $k[[x]]$. We can define an extended map

$$\operatorname{ord}_w : E \to \mathbb{Z} \cup \{\infty\}$$

by putting $\operatorname{ord}_w(h) = \operatorname{ord}_w(f) - \operatorname{ord}_w(g)$, where $h \in E$, and $f, g \in k[[x]]$ with $g \neq 0$ are elements with $h = f/g$.

If $w_i = 1$ for every $0 \leq i \leq n$, we write ord instead of ord_w. The value $\operatorname{ord}(f)$ is called the *order* of f.

Lemma 2.7. 1. *The map $\operatorname{ord}_w : E \to \mathbb{Z} \cup \{\infty\}$ associated with an element $w \in \mathbb{Z}_0^{n+1}$ and a parameter system Q is a valuation of the field E with the value group $\mathbb{Z}$.*
2. *$\operatorname{ord}_w(f) \geq 0$ for every $f \in k[[x]]$.*
3. *Assume that $w_i > 0$ for every $0 \leq i \leq n$. If $f \in k[[x]]$ and $\operatorname{ord}_w(f) = 0$, then $f \in k[[x]]^*$.*
4. *Assume $w_i = 1$ for every $0 \leq i \leq n$. Then, $\operatorname{ord}_w = \operatorname{ord}$ does not depend on the choice of the parameter system Q.*

Proof. We consider claim 4. Under the assumption we have $\operatorname{ord}_w(f) = \max\{\nu \mid f \in M(k[[x]])^\nu\}$, which shows the claim. □

Definition 2.8. Let $f \in k[[x]]$ be a non-zero element. Let $f = u g_1^{\beta_1} g_2^{\beta_2} \cdots g_r^{\beta_r}$ be the irreducible decomposition of f, where $g_1, g_2, \dots, g_r \in k[[x]]$ are irreducible elements such that any two of them are coprime, $u \in k[[x]]^*$ and $\beta_1, \beta_2, \dots, \beta_r$ are positive integers. We assume that $\operatorname{ord}(g_i) > 1$ for $1 \leq i \leq s$ and $\operatorname{ord}(g_i) = 1$ for $s + 1 \leq i \leq r$.

1. A triplet (v, h_s, h_n) satisfying the following conditions are called the *singular non-singular decomposition* of f.
   (a) $f = v h_s h_n$. $v \in k[[x]]^*$. $h_s \in k[[x]]$. $h_n \in k[[x]]$.
   (b)

$$\frac{h_s}{g_1^{\beta_1} g_2^{\beta_2} \cdots g_s^{\beta_s}} \in k[[x]]^*.$$

(c)

$$\frac{h_n}{g_{s+1}^{\beta_{s+1}} g_{s+2}^{\beta_{s+2}} \cdots g_r^{\beta_r}} \in k[[x]]^*.$$

2. Let (v, h_s, h_n) be the singular non-singular decomposition of f. We call $\mathrm{ord}(h_s)$ the *singular order* of f, and denote it by $\mathrm{sord}(f)$. We call $\mathrm{ord}(h_n)$ the *non-singular order* of f, and denote it by $\mathrm{nord}(f)$.

Lemma 2.9. 1. *The singular order* $\mathrm{sord}(f)$ *and the non-singular order* $\mathrm{nord}(f)$ *of an element* $f \in k[[x]] - \{0\}$ *are well-defined. They do not depend on the choice of the singular non-singular decomposition* (v, h_s, h_n) *of* f *in the definition.*

2. $\mathrm{sord}(f)$ *and* $\mathrm{nord}(f)$ *are non-negative integers.* $\mathrm{ord}(f) = \mathrm{sord}(f) + \mathrm{nord}(f)$. $\mathrm{sord}(f) \neq 1$. $\mathrm{sord}(fg) = \mathrm{sord}(f) + \mathrm{sord}(g)$. $\mathrm{nord}(fg) = \mathrm{nord}(f) + \mathrm{nord}(g)$.

Let $w = (w_0, w_1, \dots, w_n) \in \mathbb{Z}_0^{n+1}$, and $Q = \{x_0, x_1, \dots, x_n\}$ be a parameter system of $k[[x]]$.

Let $f = \sum_{\alpha \in \mathbb{Z}_0^{n+1}} f_\alpha x^\alpha \in k[[x]]$ be a non-zero element. Let $\mathrm{iex}(f) = \{\alpha \in \mathrm{ex}(f) \mid \sum_{i=0}^n w_i \alpha_i = \mathrm{ord}_w(f)\}$. Putting

$$\mathrm{in}_w(f) = \sum_{\alpha \in \mathrm{iex}(f)} f_\alpha x^\alpha,$$

we call $\mathrm{in}_w(f)$ the *w-initial polynomial* of f. For $f = 0 \in k[[x]]$ we define $\mathrm{in}_w(0) = 0$.

We can check that $\mathrm{in}_w(fg) = \mathrm{in}_w(f)\mathrm{in}_w(g)$ for any $f, g \in k[[x]]$.

Assume that $w_i = 0$ for $0 \le i \le m$ and $w_i > 0$ for $m+1 \le i \le n$. One sees that $\mathrm{in}_w(f) \in k[[x_0, x_1, \dots, x_m]][x_{m+1}, x_{m+2}, \dots, x_n]$.

We say that f is a *w-homogeneous polynomial*, if $f = \mathrm{in}_w(f)$. If $w_i > 0$ for every $0 \le i \le n$, then any w-homogeneous polynomial contains only a finite number of terms, and it is a polynomial with coefficients in k.

Consider the case where $w_i = 1$ for every $0 \le i \le n$ in particular. In this case $\mathrm{in}_w(f)$ does not depend on the choice of the parameter system Q. We write $\mathrm{in}(f)$ instead of $\mathrm{in}_w(f)$, and call $\mathrm{in}(f)$ the *initial polynomial* of f. It has only a finite number of terms. Also in this case a w-homogeneous polynomial is called a *homogeneous polynomial.*

In the theorem below one of $n+1$ variables plays a special role. Putting $t = x_0$, and $y_i = x_i$ for $1 \le i \le n$, we write

$$\begin{array}{rcccl} k[[t, y]] & = & k[[t, y_1, y_2, \dots, y_n]] & = & k[[x]], \\ k[[y]] & = & k[[y_1, y_2, \dots, y_n]]. & & \end{array}$$

Theorem 2.10 (Weierstrass division theorem)**.** *Let* $w = (w_0, w_1, \dots, w_n) \in \mathbb{Z}_0^{n+1}$. *We assume that* $w_i > 0$ *for every* $0 \le i \le n$. *Let* $f \in k[[t, y]]$. *Assume that* $\mathrm{in}_w(f)(1, 0, 0, \dots, 0) \neq 0$, *where* $\mathrm{in}_w(f)(1, 0, 0, \dots, 0)$ *denotes the result of substitution* $t = 1$, $y_1 = y_2 = \cdots = y_n = 0$ *in the w-initial polynomial of* f.

1. *The quotient* $m = \mathrm{ord}_w(f)/\mathrm{ord}_w(t)$ *is a non-negative integer.*
2. *For every* $h \in k[[t, y]]$ *there exists a unique set of an element* $e \in k[[t, y]]$ *and* m *numbered elements* $H_1, H_2, \dots, H_m \in k[[y]]$ *satisfying*

$$h = ef + \sum_{j=1}^m H_j t^{m-j}.$$

Proof. Not difficult. □

Corollary 2.11 (Weierstrass preparation theorem)**.** *Let* $w = (w_0, w_1, \dots, w_n) \in \mathbb{Z}_0^{n+1}$*. We assume that* $w_i > 0$ *for every* $0 \leq i \leq n$*. Let* $f \in k[[t, y]]$*. Assume that* $\mathrm{in}_w(f)(1, 0, 0, \dots, 0) \neq 0$*. Let* $m = \mathrm{ord}_w(f)/\mathrm{ord}_w(t)$*, which is a non-negative integer.*

1. *There exists a unique set of an element* $e \in k[[t, y]]$ *and* m *numbered elements* $G_1, G_2, \dots, G_m \in k[[y]]$ *satisfying*
$$ef = t^m + \sum_{j=1}^{m} G_j t^{m-j}.$$
2. $e \in k[[t, y]]^*$*, and* $\mathrm{ord}_w(G_j) \geq j \,\mathrm{ord}_w(t)$ *for every* $1 \leq j \leq m$*.*
3. *The element* $g = t^m + \sum_{j=1}^{m} G_j t^{m-j} \in k[[t, y]]$ *belongs to the subring* $k[[y]][t]$*, or the ring of polynomials with variable* t *with coefficients in* $k[[y]]$*.* $\mathrm{ord}_w(g) = m \,\mathrm{ord}_w(t)$*.*

Proof. It follows from the special case of Theorem 2.10 where $h = t^m$. ☐

Lemma 2.12. *Let* $f \in k[[x]] = k[[x_0, x_1, \dots, x_n]]$ *be a non-zero element.*

1. *There exist* n *elements* $a_1, a_2, \dots, a_n \in k$ *with* $\mathrm{in}(f)(1, a_1, a_2, \cdots, a_n) \neq 0$*, where* $\mathrm{in}(f)(1, a_1, a_2, \cdots, a_n)$ *denotes the result of substitution* $x_0 = 1, x_1 = a_1, x_2 = a_2, \dots, x_n = a_n$ *in* $\mathrm{in}(f)$*.*
2. *Assume that* $\mathrm{in}(f)(1, a_1, a_2, \cdots, a_n) \neq 0$ *for* $a_1, a_2, \dots, a_n \in k$*. Let* $t = x_0$ *and* $y_i = x_i - a_i x_0$ *for* $1 \leq i \leq n$*. Then* $t, y_1, y_2, \dots, y_n$ *is a parameter system of* $k[[x]]$*, and the result of substitution* $t = 1, y_1 = y_2 = \cdots = y_n = 0$ *in* $\mathrm{in}(f)$ *is not zero.*

Proposition 2.13. *Let* $f_1, f_2, \dots, f_\ell \in k[[x]] = k[[x_0, x_1, \dots, x_n]]$ *be a finite number of elements with* $f = f_1 f_2 \cdots f_\ell \neq 0$*. Let* $m_\nu = \mathrm{ord}(f_\nu)$ *for* $1 \leq \nu \leq \ell$*.*

Let $a_1, a_2, \dots, a_n \in k$ *be elements with* $\mathrm{in}(f)(1, a_1, a_2, \dots, a_n) \neq 0$*.*

Let $t, y_1, y_2, \dots, y_n$ *be the parameter system of* $k[[x]]$ *satisfying* $t = x_0$ *and* $y_i = x_i - a_i x_0$ *for* $1 \leq i \leq n$*. We denote* $k[[t, y]] = k[[t, y_1, y_2, \dots, y_n]] = k[[x]]$ *and* $k[[y]] = k[[y_1, y_2, \dots, y_n]]$*. For every* $1 \leq \nu \leq \ell$ *the following* 1–3 *hold:*

1. *The result of substitution* $t = 1$*,* $y_1 = y_2 = \cdots = y_n = 0$ *in the initial polynomial of* f_ν *is not equal to* 0*.*
2. *There exists a unique set of an element* $e_\nu \in k[[t, y]]$ *and* m_ν *numbered elements* $G_{\nu,1}, G_{\nu,2}, \dots, G_{\nu,m_\nu} \in k[[y]]$ *satisfying* $e_\nu f_\nu = t^{m_\nu} + \sum_{j=1}^{m_\nu} G_{\nu,j} t^{m_\nu - j}$*.*
3. $e_\nu \in k[[t, y]]^*$*, and* $\mathrm{ord}(G_{\nu,j}) \geq j$ *for every* $1 \leq j \leq m_\nu$*.*

Proof. It follows from Lemma 2.12 and Corollary 2.11. ☐

Lemma 2.14. *Let* $f \in k[[x_0, x_1, \dots, x_n]]$*. The following three conditions are equivalent:*

1. $f \in k[[x_0^p, x_1^p, \dots, x_n^p]]$*.*
2. *There exists* $g \in k[[x_0, x_1, \dots, x_n]]$ *with* $f = g^p$*.*
3. $\frac{\partial f}{\partial x_i} = 0$ *for every* $0 \leq i \leq n$*.*

Lemma 2.15. *Let* $f \in k[[x]] = k[[x_0, x_1, \dots, x_n]]$ *be an irreducible element. We define a* k*-linear map*
$$\lambda : k^{n+1} \to k[[x]]/fk[[x]]$$
by $\lambda(a_0, a_1, a_2, \dots, a_n) = \sum_{i=0}^{n} a_i(\partial f/\partial x_i) \mod fk[[x]]$*.*

1. *The map* λ *is not zero.*

2. *There is a homogeneous polynomial h of degree* 1 *whose zero-locus contains the kernel of λ.*

Proof. 1. By Lemma 2.12 we have $\mathrm{in}(f)(1,0,0,\cdots,0) \neq 0$ after a linear coordinate change. By Corollary 2.11 we have an invertible element $e \in k[[x]]^*$ and m numbered elements $G_1, G_2, \dots, G_m \in k[[x_1, x_2, \dots, x_n]]$ where $m = \mathrm{ord}(f)$ satisfying

$$ef = g, \; g = x_0^m + \sum_{j=1}^{m} G_j x_0^{m-j}.$$

Assume that λ is zero. For every $0 \leq i \leq n$ we have

$$\frac{\partial g}{\partial x_i} = \frac{\partial e}{\partial x_i} f + e \frac{\partial f}{\partial x_i} \in fk[[x]].$$

Thus, there is $e_1 \in k[[x]]$ with $\frac{\partial g}{\partial x_i} = e_1 f$.

On the other hand, we can write

$$\frac{\partial g}{\partial x_i} = \begin{cases} 0 \cdot f + m x_0^{m-1} + \sum_{j=2}^{m} (m-j+1) G_{j-1} x_0^{m-j} & \text{if } i = 0, \\ 0 \cdot f + \sum_{j=1}^{m} \frac{\partial G_j}{\partial x_i} x_0^{m-j} & \text{if } 1 \leq i \leq n. \end{cases}$$

By Theorem 2.10.2 we have $e_1 = 0$ and $\frac{\partial g}{\partial x_i} = 0$. By Lemma 2.14 there is $g_1 \in k[[x]]$ with $g = g_1^p$. Thus $f = g_1^p/e$ is not irreducible, which contradicts the assumption.
2. Obvious. □

Definition 2.16. 1. Assume that a parameter system Q of the ring $k[[x]]$ is given. We call a pair $(g,\, t)$ of elements $g \in k[[x]]$ and $t \in Q$ a *semi-Weierstrass pair* of $k[[x]]$ *under* Q, if there exist a non-negative integer m and m elements $G_1, G_2, \dots, G_m \in k[[y]] = k[[y_1, y_2, \dots, y_n]]$ satisfying $g = t^m + \sum_{j=1}^{m} G_j t^{m-j}$, where $y_1, y_2, \dots,\, y_n$ are elements in Q except t.

2. A non-empty subset of a parameter system of a regular local ring is called a *partial parameter system.*
3. Assume that a partial parameter system P of the ring $k[[x]]$ with $\sharp P = \dim k[[x]] - 1$ is given. We call a pair $(g,\, t)$ of elements $g \in k[[x]]$ and $t \in k[[x]]$ a *semi-Weierstrass pair* of $k[[x]]$ *over* P, if $t \notin P$ and the union $P \cup \{t\}$ is a parameter system of $k[[x]]$, and if there exist a non-negative integer m and m elements $G_1, G_2, \dots, G_m \in k[[y]] = k[[y_1, y_2, \dots, y_n]]$ satisfying $g = t^m + \sum_{j=1}^{m} G_j t^{m-j}$, where $P = \{y_1, y_2, \dots, y_n\}$.
4. The integer m is called the *degree* of $(g,\, t)$, and is denoted by $\deg(g,\, t)$.
5. The m-tuple $(G_1, G_2, \dots, G_m)$ of elements in $k[[y]]$ is called the *coefficient functions* of $(g,\, t)$.
6. If $m = \deg(g,\, t) = 0$, we define $\mathrm{w}(g,\, t) = \infty$. If $m = \deg(g,\, t) > 0$, we define $\mathrm{w}(g,\, t) = \min\{\mathrm{ord}(G_j)/j \mid 1 \leq j \leq m\}$. We call $\mathrm{w}(g,\, t) \in \mathbb{Q} \cup \{\infty\}$ the *weight* of $(g,\, t)$.
7. For $1 \leq j \leq m = \deg(g,\, t)$, let

$$\overline{G_j} = \begin{cases} 0 & \text{if } \mathrm{ord}(G_j) > j\,\mathrm{w}(g,\, t) \\ \mathrm{in}(G_j) & \text{if } \mathrm{ord}(G_j) = j\,\mathrm{w}(g,\, t). \end{cases}$$

The polynomial $t^m + \sum_{j=1}^{m} \overline{G_j} t^{m-j}$ is called the *initial polynomial* of $(g,\, t)$, and is denoted by $\mathrm{in}(g,\, t)$.

8. If the coefficient functions $(G_1, G_2, \dots, G_m)$ of a semi-Weierstrass pair $(g,\, t)$ satisfies $\mathrm{ord}(G_j) \geq j$ for every $1 \leq j \leq m$, then we say that the pair $(g,\, t)$ is a *Weierstrass pair.*

Lemma 2.17. *Assume either that a parameter system Q of the ring $k[[x]]$ is given, or that a partial parameter system P of the ring $k[[x]]$ with $\sharp P = \dim k[[x]] - 1$ is given. Let $(g,\, t)$ be a semi-Weierstrass pair either under Q or over P.*

1. *The degree $\deg(g,\, t)$, the coefficient functions $(G_1,\, G_2, \dots, G_m)$, the weight $\mathrm{w}(g,\, t)$ and the initial polynomial $\mathrm{in}(g,\, t)$ of $(g,\, t)$ are uniquely defined depending on the pair $(g,\, t)$.*
2. $\deg(g,\, t) \geq \mathrm{ord}(g)$.
3. *Let $m = \deg(g,\, t)$. The following three conditions are equivalent:*
   (a) $\mathrm{w}(g,\, t) = \infty$.
   (b) $G_j = 0$ *for every* $1 \leq j \leq m$.
   (c) $\mathrm{in}(g,\, t) = t^m$.
4. *The following three conditions are equivalent:*
   (a) *The pair $(g,\, t)$ is a Weierstrass pair.*
   (b) $\deg(g,\, t) = \mathrm{ord}(g)$.
   (c) $\mathrm{w}(g,\, t) \geq 1$.

Lemma 2.18. *Let Q be a parameter system of the ring $k[[x]]$, and $(g,\, t)$ and $(g',\, t)$ semi-Weierstrass pairs under Q with the common second term $t \in Q$. Let D denote the field of fractions of $k[[y]] = k[[y_1, y_2, \dots, y_n]]$ where $y_1, y_2, \dots, y_n$ are elements in Q except t. Let $g = g_1^{\beta_1} g_2^{\beta_2} \cdots g_r^{\beta_r}$ be the irreducible decomposition in the ring $D[t]$. We assume that the coefficient of the term with the highest degree in $g_\nu \in D[t]$ is equal to 1 for every $1 \leq \nu \leq r$.*

1. *The pair $(gg',\, t)$ is a semi-Weierstrass pair.* $\deg(gg',\, t) = \deg(g,\, t) + \deg(g',\, t)$. *The pair $(gg',\, t)$ is a Weierstrass pair, if and only if, both of $(g,\, t)$ and $(g',\, t)$ are Weierstrass pairs.*
2. *For every $1 \leq \nu \leq r$, $g_\nu \in k[[y]][t] \subset k[[x]]$.*
3. *Assume that $(g,\, t)$ is a Weierstrass pair. For every $1 \leq \nu \leq r$, $(g_\nu,\, t)$ is a Weierstrass pair, and the decomposition $g = g_1^{\beta_1} g_2^{\beta_2} \cdots g_r^{\beta_r}$ gives the irreducible decomposition in the ring $k[[x]]$.*

Proof. The claim 2 follows from Lemma of Gauss. (Zariski-Samuel[10], Vol.I, p.32.) □

Definition 2.19. Let P be a partial parameter system of the ring $k[[x]]$, and $(g,\, t)$ a semi-Weierstrass pair over P. Let D denote the field of fractions of $k[[y]] = k[[y_1, y_2, \dots, y_n]]$ where $P = \{y_1, y_2, \dots, y_n\}$. Let $g = g_1^{\beta_1} g_2^{\beta_2} \cdots g_r^{\beta_r}$ be the irreducible decomposition in the ring $D[t]$. A semi-Weierstrass pair $(g,\, t)$ is said to be *separable*, if g_ν is a separable polynomial in $D[t]$ for every $1 \leq \nu \leq r$.

Note Lemma 2.15.

Lemma 2.20. *Let $f \in k[[x]]$ be a non-zero element, and $f = u f_1^{\alpha_1} f_2^{\alpha_2} \cdots f_r^{\alpha_r}$ ($u \in k[[x]]^*$) the irreducible decomposition of f. For every $1 \leq \nu \leq r$ we define a k-linear map*

$$\lambda_\nu : k^{n+1} \to k[[x]]/f_\nu k[[x]]$$

by $\lambda_\nu(a_0, a_1, a_2, \dots, a_n) = \sum_{i=0}^{n} a_i(\partial f_\nu/\partial x_i) \mod f_\nu k[[x]]$. Let h_ν be a homogeneous polynomial of degree 1 whose zero-locus contains the kernel of λ_ν. Let $\Psi = \mathrm{in}(f) \prod_{\nu=1}^{r} h_\nu$.

1. *There exists elements* $a_1, a_2, \ldots, a_n \in k$ *with* $\Psi(1, a_1, a_2, \ldots, a_n) \neq 0$.
2. *Assume* $a_1, a_2, \ldots, a_n \in k$ *and* $\Psi(1, a_1, a_2, \ldots, a_n) \neq 0$. *Let* $t = x_0$, $y_i = x_i - a_i x_0$ *for* $1 \leq i \leq n$, *and* $Q = \{t, y_1, y_2, \ldots .y_n\}$. *There is an invertible element* $e \in k[[x]]^*$ *such that* (ef, t) *is a separable Weierstrass pair under the parameter system* Q.

Proof. 1. It follows from that Ψ is a non-zero homogeneous polynomial.
2. We have $\mathrm{in}(f) = u(0)\mathrm{in}(f_1)^{\alpha_1}\mathrm{in}(f_2)^{\alpha_2} \cdots \mathrm{in}(f_r)^{\alpha_r}$. Thus, $\Psi(1, a_1, a_2, \ldots, a_n) \neq 0$, if and only if, $\mathrm{in}(f_\nu)^{\alpha_\nu}(1, a_1, a_2, \ldots, a_n) \neq 0$ and $h_\nu(1, a_1, a_2, \ldots, a_n) \neq 0$ for every $1 \leq \nu \leq r$. By definition $h_\nu(1, a_1, a_2, \ldots, a_n) \neq 0$ implies $\lambda_\nu(1, a_1, a_2, \ldots, a_n) \neq 0$.

Let ν be an arbitrary integer with $1 \leq \nu \leq r$. By Lemma 2.12.2 and by Corollary 2.11 we have an invertible element $e_\nu \in k[[x]]^*$ such that $(e_\nu f_\nu, t)$ is a Weierstrass pair under Q. Let $g_\nu = e_\nu f_\nu \in D[t]$. We have

$$\begin{aligned}
\frac{\partial g_\nu}{\partial t} &= \frac{\partial e_\nu}{\partial t} f_\nu + e_\nu \frac{\partial f_\nu}{\partial t} \\
&= \frac{\partial e_\nu}{\partial t} f_\nu + e_\nu \sum_{i=0}^{\nu} \frac{\partial x_i}{\partial t} \frac{\partial f_\nu}{\partial x_i} \\
&= \frac{\partial e_\nu}{\partial t} f_\nu + e_\nu \left(\frac{\partial f_\nu}{\partial x_0} + \sum_{i=1}^{\nu} a_i \frac{\partial f_\nu}{\partial x_i} \right).
\end{aligned}$$

Since $0 \neq \lambda_\nu(1, a_1, a_2, \ldots, a_n) = \frac{\partial f_\nu}{\partial x_0} + \sum_{i=1}^{\nu} a_i \frac{\partial f_\nu}{\partial x_i} \mod f_\nu k[[x]]$, One has $\frac{\partial g_\nu}{\partial t} \notin f_\nu k[[x]]$. In particular, $\frac{\partial g_\nu}{\partial t} \neq 0$, and g_ν is a separable polynomial in $D[t]$. Since $f_\nu = g_\nu / e_\nu$ is irreducible in $k[[x]]$, by Lemma 2.18.3 g_ν is irreducible in $D[t]$.

Let $e = e_1^{\alpha_1} e_1^{\alpha_1} \cdots e_r^{\alpha_r}/u \in k[[x]]^*$. We have $ef = g_1^{\alpha_1} g_1^{\alpha_1} \cdots g_r^{\alpha_r}$. The pair (ef, t) is a Weierstrass pair under Q by Lemma 2.18.1. Since the irreducible decomposition in $D[t]$ is unique, $ef = g_1^{\alpha_1} g_1^{\alpha_1} \cdots g_r^{\alpha_r}$ gives the irreducible decomposition in $D[t]$. Thus (ef, t) is separable. ☐

3. Space germ

In this section we give the definition of our main objects, a space germ, a framed space germ, a basic space germ, an elementary monoidal transform, an iterated elementary monoidal transform (an IEMT, in short), an iterated analytic monoidal transform (an IAMT, in short) and so on.

Let $k[[x]] = k[[x_0, x_1, \ldots, x_r]]$ denote the formal power series ring over k with $r + 1$ variables $x_0, x_1, \ldots, x_r$. We give the $M(k[[x]])$-adic topology to the ring $k[[x]]$. Recall that a non-empty subset of a parameter system of a regular local ring is called a *partial parameter system.* (Definition 2.16.2.)

Definition 3.1. 1. A ring B is called a *space germ* over k, if B is a topological local k-algebra isomorphic to $k[[x]] = k[[x_0, x_1, \ldots, x_r]]$ for some non-negative integer r.

2. A partial parameter system Q of B is called a *frame* of B.
3. For any partial parameter system $Q = \{y_0, y_1, \ldots, y_n\}$ of B by $k[[Q]]$ we denote the minimum closed k-subalgebra $k[[y_0, y_1, \ldots, y_n]]$ of B generated by $y_0, y_1, \cdots, y_n$.
4. A pair (B, Q), where B is a space germ over k, and Q is a frame of B, is called a *framed space germ* over k.

5. The number $\dim B = r+1$ is called the *dimension* of a space germ B or of a framed space germ (B, Q).
6. The number $\sharp\, Q$ of elements in Q is called the *basic dimension* of a framed space germ (B, Q).

Lemma 3.2. *Let (B, Q) be a framed space germ. Let $A = k[[Q]]$. The pair (A, Q) is a framed space germ over k such that Q is a parameter system of A.*

Definition 3.3. 1. Let (B, Q) be a framed space germ. The space germ B without a frame is called the *underlying space germ* of (B, Q).

2. We call the framed space germ (A, Q) described in Lemma 3.2 the *basic framed space germ* of a framed space germ (B, Q).
3. Let B and B' be space germs. A continuous local homomorphism $\varphi : B \to B'$ of k-algebras is called a *morphism* of space germs from B to B'.
4. Let (B, Q) and (B', Q') be framed space germs. A continuous local morphism $\varphi : B \to B'$ of k-algebras is called a *morphism* of framed space germs from (B, Q) to (B', Q'), if the following conditions are satisfied:
   (a) $\varphi(k[[Q]]) \subset k[[Q']]$, and the induced homomorphism $\varphi : k[[Q]] \to k[[Q']]$ are local and continuous.
   (b) There exists a parameter system R of B such that $Q \subset R$, $\varphi(R-Q) \cap Q' = \emptyset$, and $\varphi(R-Q) \cup Q'$ is a parameter system of B'.

   We denote $\varphi : (B, Q) \to (B', Q')$.

Lemma 3.4. *Let (B, Q) be a framed space germ, and $A = k[[Q]]$.*

1. *The inclusion homomorphism $A \subset B$ defines a morphism $A \to B$ of space germs.*
2. *Let $R = \{x_0, x_1, \dots, x_r\}$ be a parameter system of B containing the parameter system $Q = \{x_0, x_1, \dots, x_n\}$ of A. Let $R' = \{y_0, y_1, \dots, y_s\}$ be a finite subset of B containing a parameter system $Q' = \{y_0, y_1, \dots, y_n\}$ of A. Then, R' is a parameter system of B, if and only if, $s = r$, $R' \subset M(B)$ and* $\det(\partial y_j / \partial x_i)_{n<i,\, j \le r} \notin M(B)$.

We call the morphism $A \to B$ in Lemma 3.4.1 the *inclusion morphism.*

Lemma 3.5. *Let $\varphi : (B, Q) \to (B', Q')$ be a morphism of framed space germs. Let $A = k[[Q]]$, and $A' = k[[Q']]$.*

1. *For any parameter system R_1 of B containing a parameter system Q_1 of A and for any parameter system Q'_1 of A', the union $\varphi(R_1 - Q_1) \cup Q'_1$ is a parameter system of B'.*
2. *The morphism φ induces a morphism $\varphi_\bullet : B \to B'$ of underlying space germs.*
3. *The morphism φ induces a morphism $\underline{\varphi} : (A, Q) \to (A', Q')$ of basic framed space germs.*

Proof. 1. It follows from Lemma 3.4.2. □

We call $\varphi_\bullet$ the *underlying morphism* of φ, while we call $\underline{\varphi}$ the *basic morphism* of φ.

Lemma 3.6. *Let $\varphi : (B, Q) \to (B', Q')$ be a morphism of framed space germs. Let Q_1 be a partial parameter system of B with $Q \subset Q_1$.*

Let $Q'_1 = Q' \cup \varphi(Q_1 - Q)$. The set Q'_1 is a partial parameter system of B'. It satisfies $Q' \subset Q'_1$ and $\sharp\,(Q_1 - Q) = \sharp\,(Q'_1 - Q')$. There exists a unique morphism

$\varphi_1 : (B, Q_1) \to (B', Q_1')$ of framed space germs such that $\varphi_\bullet = (\varphi_1)_\bullet$ and the following diagram is commutative:

$$\begin{tikzcd}
k[[Q]] \arrow[r] \arrow[d, "\underline{\varphi}_\bullet"] & k[[Q_1]] \arrow[r] \arrow[d, "(\underline{\varphi}_1)_\bullet"] & B \arrow[d, "\varphi_\bullet=(\varphi_1)_\bullet"] \\
k[[Q']] \arrow[r] & k[[Q_1']] \arrow[r] & B'
\end{tikzcd}$$

Proof. The claim on Q_1' follows from Lemma 3.5.1. The remaining parts are easy. □

Definition 3.7. Let (B, Q) be a framed space germ, and Q' an arbitrary parameter system of $k[[Q]]$. The pair (B, Q') is a framed space germ, and we call it a *coordinate transform* of (B, Q).

The morphism $(B, Q) \to (B, Q')$ of space germs induced by the identity homomorphisms of B is called a *coordinate transformation.*

Note that the morphism $(B, Q) \to (B, Q')$ can be defined above, since for any parameter system R of B containing Q, the union $(R - Q) \cup Q'$ is a parameter system of B containing Q'. (Lemma 3.4.2.)

We will define an elementary monoidal transform of a framed space germ.

Let (B, Q) be a framed space germ. Let $C \subset Q$ be a subset containing at least two elements, $x_0 \in C$ an arbitrary element, and $\lambda : C - \{x_0\} \to k$ an arbitrary map from $C - \{x_0\}$ to k. By $\sigma : \Sigma \to \mathrm{Spec}(B)$ we denote the blowing-up with center in the non-singular closed subscheme of $\mathrm{Spec}(B)$ defined by the ideal generated by elements in C. We denote $C = \{x_0, x_1, \ldots, x_t\}$ and $\lambda_i = \lambda(x_i) \in k$ for $1 \le i \le t$.

Consider a B-subalgebra

$$B_1 = B[\frac{x_1}{x_0}, \frac{x_2}{x_0}, \cdots, \frac{x_t}{x_0}]$$

generated by t elements $x_1/x_0, x_2/x_0, \cdots, x_t/x_0$ in the field of fractions of B. The affine scheme $\mathrm{Spec}(B_1)$ is a coordinate chart of Σ, and $t{+}1$ affine schemes $\mathrm{Spec}(B_1)$'s obtained when x_0 runs on C covers Σ. Let $\mathfrak{m}$ be the ideal in B_1 generated by the maximal ideal $M(B)$ of B and t elements $\frac{x_i}{x_0} - \lambda_i$, $1 \le i \le t$. We have $B_1/\mathfrak{m} \cong k$ and $\mathfrak{m} \cap B = M(B)$. Thus $\mathfrak{m}$ is a maximal ideal of B_1, and defines a closed point $s \in \mathrm{Spec}(B_1) \subset \Sigma$ lying over the unique closed point of $\mathrm{Spec}(B)$. Let $B_2 = (B_1)_{\mathfrak{m}}$. One has $\mathcal{O}_{\Sigma,s} = B_2$. Let $y_0 = x_0$ and $y_i = (x_i/x_0) - \lambda_i$ for $1 \le i \le t$, and $Q' = (Q - C) \cup \{y_0, y_1, \ldots, y_t\}$. One sees that B_2 is a regular local ring, and Q' is a frame of B_2.

Let B' be the completion of B_2. Ring B' is a complete regular local ring and Q' is a frame of B'. Ring B' coincides with the completion of B_1 with respect to the ideal $\mathfrak{m}$. The inclusion homomorphism $B \subset B_2$ and the canonical homomorphism $B_2 \to B'$ to the completion define an injective continuous homomorphism $B \to B'$ of k-algebras.

The pair (B', Q') is a framed space germ over k.

Let $A = k[[Q]]$, $A' = k[[Q']]$ $A_1 = A[x_1/x_0,\ x_2/x_0, \ldots, x_t/x_0]$, and $\mathfrak{l} = A_1 \cap \mathfrak{m}$. One sees easily that the completion of A_1 with respect to $\mathfrak{l}$ coincides with A'. Thus composing the inclusion homomorphism $A \to A_1$ and the canonical homomorphism $A_1 \to A'$ to the completion, we obtain an injective continuous homomorphism $A \to A'$ of k-algebras. Obviously $A \to A'$ coincides with the restriction of the homomorphism $B \to B'$ to A.

The homomorphism $B \to B'$ induces a morphism $(B, Q) \to (B', Q')$ of framed space germs.

Definition 3.8. Let (B, Q) be a framed space germ over k.

1. A triplet (C, x_0, λ) satisfying the following three conditions is called a *datum of a monoidal transformation* over (B, Q).
   (a) $C \subset Q$. C contains at least two elements.
   (b) $x_0 \in C$.
   (c) $\lambda : C - \{x_0\} \to k$ is a map.
2. Let (C, x_0, λ) be a datum of a monoidal transformation over (B, Q). Let B' denote the ring described above. For every $x \in C$ with $x \neq x_0$ we set $y_x = (x/x_0) - \lambda(x)$. Let $Q' = (Q - C) \cup \{x_0\} \cup \{y_x \mid x \in C, x \neq x_0\}$. The framed space germ (B', Q') is called the *elementary monoidal transform* of (B, Q) associated with a datum (C, x_0, λ).
3. The induced morphism $(B, Q) \to (B', Q')$ described above is called the *monoidal transformation* associated with the elementary monoidal transform (B', Q').
4. A space germ B' without a frame is called an elementary monoidal transform of B, if there exist a frame Q of B and a datum (C, x_0, λ) of a monoidal transformation over (B, Q) such that the underlying space germ of the elementary monoidal transform (B', Q') of (B, Q) associated with (C, x_0, λ) coincides with B'. The underlying morphism $B \to B'$ of the monoidal transformation $(B, Q) \to (B', Q')$ is called the monoidal transformation associated with the elementary monoidal transform B' of B.

Definition 3.9. Let (B, Q) and (B', Q') be framed space germs over k. We consider a finite sequence $\{(B_\nu, Q_\nu) \mid 0 \leq \nu \leq \omega\}$ of framed space germs and conditions below.

1. $(B, Q) = (B_0, Q_0)$, $(B', Q') = (B_\omega, Q_\omega)$.
2. For every number ν with $1 \leq \nu \leq \omega$, (B_ν, Q_ν) is an elementary monoidal transform of $(B_{\nu-1}, Q_{\nu-1})$ associated with some datum.
3. For every number ν with $1 \leq \nu \leq \omega$, (B_ν, Q_ν) is an elementary monoidal transform of $(B_{\nu-1}, Q_{\nu-1})$ associated with some datum, or a coordinate transform of $(B_{\nu-1}, Q_{\nu-1})$.)

If there is a sequence $\{(B_\nu, Q_\nu) \mid 0 \leq \nu \leq \omega\}$ satisfying 1 and 2, then we say that (B', Q') is an *iterated elementary monoidal transform*, or an *IEMT* in short, of (B, Q). If there is a sequence $\{(B_\nu, Q_\nu) \mid 0 \leq \nu \leq \omega\}$ satisfying 1 and 3, then we say that (B', Q') is an *iterated analytic monoidal transform*, or an *IAMT* in short, of (B, Q). The composition $(B, Q) = (B_0, Q_0) \to (B_\omega, Q_\omega) = (B', Q')$ of ω associated morphisms $(B_{\nu-1}, Q_{\nu-1}) \to (B_\nu, Q_\nu)$, $1 \leq \nu \leq \omega$ is called the *iterated monoidal transformation*, or *IMT* in short, associated with the IEMT or IAMT (B', Q') of (B, Q).

A space germ B' without a frame is called an IAMT of B, if there exist a frame Q of B and an IAMT (B', Q') of (B, Q) such that the underlying space germ of (B', Q') coincides with B'. The underlying morphism $B \to B'$ of the IMT $(B, Q) \to (B', Q')$ is called the iterated monoidal transformation associated with the IAMT B' of B.

Definition 3.10. Let (B, Q) be a framed space germ over k, and (C, x_0, λ) and (C', x'_0, λ') data of a monoidal transformation over (B, Q). We say that (C, x_0, λ) and (C', x'_0, λ') are *equivalent*, if one of the following two conditions holds:

1. $(C, x_0, \lambda) = (C', x'_0, \lambda')$
2. $C = C'$, $x_0 \neq x'_0$, $\lambda(x'_0)\lambda'(x_0) = 1$, and $\lambda(y) = \lambda'(y)/\lambda'(x_0)$ for every $y \in C - \{x_0, x'_0\}$.

Lemma 3.11. 1. *The relation of data of a monoidal transformation defined in Definition 3.10 is an equivalence relation.*

2. *Let* (C, x_0, λ) *and* (C', x'_0, λ') *be data of a monoidal transformation over a framed space germ* (B, Q) *over* k. *Assume* $C = C'$. *By* $\sigma : \Sigma \to \mathrm{Spec}(B)$ *we denote the blowing-up with center in the non-singular closed subscheme of* $\mathrm{Spec}(B)$ *defined by the ideal generated by elements in* C.

   Let B_1 *be the* B*-subalgebra generated by elements* y/x_0, $y \in C - \{x_0\}$ *in the field of fractions of* B, *and* $\mathfrak{m}$ *the maximal ideal in* B_1 *generated by the maximal ideal* $M(B)$ *of* B *and elements* $(y/x_0) - \lambda(y)$, $y \in C - \{x_0\}$. *Let* B'_1 *be the* B*-subalgebra generated by elements* y/x'_0, $y \in C - \{x'_0\}$ *in the field of fractions of* B, *and* $\mathfrak{m}'$ *the maximal ideal in* B'_1 *generated by the maximal ideal* $M(B)$ *of* B *and elements* $(y/x'_0) - \lambda'(y)$, $y \in C - \{x'_0\}$.

   Data (C, x_0, λ) *and* (C, x'_0, λ') *are equivalent, if and only if, the closed point* Σ *corresponding to the maximal ideal* $\mathfrak{m}$ *and that corresponding to the maximal ideal* $\mathfrak{m}'$ *are equal.*

Definition 3.12. 1. Let $\varphi : (B, Q) \to (B', Q')$ be a morphism of framed space germs. If, for every $x \in Q$, the image $\varphi(x) \in k[[Q']]$ has a normal crossing with respect to Q', then, we say that the morphism φ *has a normal crossing.*

2. Let (B', Q') be an IAMT of a framed space germ (B, Q). We say that (B', Q') is an IAMT *with a normal crossing* of (B, Q), if the IMT $(B, Q) \to (B', Q')$ has a normal crossing.

Lemma 3.13. 1. *If a morphism* $\varphi : (B, Q) \to (B', Q')$ *has a normal crossing, then, for any parameter system* R *of* B *containing* Q *and for every element* $f \in B$ *with a normal crossing with respect to* R, $\varphi(f) \in B'$ *has a normal crossing with respect to the parameter system* $\varphi(R - Q) \cup Q'$.

2. *If two morphisms* $\varphi : (B, Q) \to (B', Q')$ *and* $\psi : (B', Q') \to (B'', Q'')$ *of framed space germs have a normal crossing, then the composition* $\psi\varphi : (B, Q) \to (B'', Q'')$ *also has a normal crossing.*
3. *The IMT associated with an IEMT has a normal crossing.*

Proof. 1, 2. Easy.
3. By 2 we can consider only the case of an elementary monoidal transform.

Assume that $y_0, y_1, \dots, y_t$ is a subset of a parameter system R' of some complete regular local ring. Assume moreover that elements $x_0, x_1, \dots, x_t$ satisfy relations $y_0 = x_0$ and $y_i = (x_i/x_0) - \lambda_i$ for $1 \leq i \leq t$, where λ_i's are elements in k.

Then, we have $x_0 = y_0$ and $x_i = y_0(\lambda_i + y_i)$ for $1 \leq i \leq t$. One sees that every x_i has a normal crossing with respect to R'.

The case of an elementary monoidal transform follows from this observation. □

We can explain our goal of this paper here.

Let (B, Q) be a framed space germ over k. Assume that a positive integer ℓ and non-zero ℓ elements $f_1, f_2, \dots, f_\ell \in k[[Q]]$ are given.

We consider the *first framed resolution game* corresponding to $f_1, f_2, \dots, f_\ell$. It is a variation of the game in INTRODUCTION. It has two players $\langle I \rangle$ and $\langle II \rangle$, and they repeat steps under the same rule. At each step, if the product $f_1 f_2 \cdots f_\ell \in k[[Q]]$

has a normal crossing with respect to Q, and if the ideal in $k[[Q]]$ generated by any non-empty subset of $\{f_1, f_2, \dots, f_\ell\}$ is a principal ideal, then the player $\langle I\rangle$ wins and they terminate the game. Otherwise, $\langle I\rangle$ has two choices. In the first choice, $\langle I\rangle$ chooses a subset $C \subset Q$ with $\sharp C \geq 2$. Then, the player $\langle II\rangle$ chooses an element $x_0 \in C$ and a map $\lambda : C - \{x_0\} \to k$. Let (B', Q') be the elementary monoidal transform of (B, Q) associated with the datum (C, x_0, λ), and by f'_i for $1 \leq i \leq \ell$ we denote the image of f_i by the monoidal transformation $B \to B'$. After replacing the pair $((B, Q), (f_1, f_2, \dots, f_\ell))$ by the pair $((B', Q'), (f'_1, f'_2, \dots, f'_\ell))$, they proceed to the next step. In the second choice, $\langle I\rangle$ chooses a parameter system Q_1 of $k[[Q]]$. After replacing the pair $((B, Q), (f_1, f_2, \dots, f_\ell))$ by the pair $((B, Q_1), (f_1, f_2, \dots, f_\ell))$, they proceed to the next step.

We consider the following statement $RS(B, Q)$ for (B, Q):

$\mathbf{RS(B, Q)}$: For any positive integer ℓ and for any non-zero ℓ elements $f_1, f_2, \dots, f_\ell$ in $k[[Q]]$, the player $\langle I\rangle$ can always win the above game.

Also we consider another statement $RW(B, Q)$ for (B, Q):

$\mathbf{RW(B, Q)}$: When $\ell = 1$, for any non-zero element $f_1 \in k[[Q]]$, the player $\langle I\rangle$ can always win the above game.

Obviously $RS(B, Q)$ implies $RW(B, Q)$. Also the converse holds. The following lemma can be shown easily:

Lemma 3.14 (Bierstone and Milman [3], p.25, Lemma 4.7). *Let (B', Q') be a framed space germ over k, and $Q' = \{y_0, y_1, \dots, y_n\}$. Let $a, b, c \in k[[Q']]^*$, and $\alpha_0, \alpha_1, \dots, \alpha_n$, $\beta_0, \beta_1, \dots, \beta_n$, $\gamma_0, \gamma_1, \dots, \gamma_n$ be non-negative integers. If*

$$ay_0^{\alpha_0} y_1^{\alpha_1} \cdots y_n^{\alpha_n} - by_0^{\beta_0} y_1^{\beta_1} \cdots y_n^{\beta_n} = cy_0^{\gamma_0} y_1^{\gamma_1} \cdots y_n^{\gamma_n},$$

then either $\alpha_i \leq \beta_i$ for every $0 \leq i \leq n$, or $\alpha_i \geq \beta_i$ for every $0 \leq i \leq n$.

Let $f_1, f_2, \dots, f_\ell \in k[[Q]]$ be finite elements as in $RS(B, Q)$. We can assume that any two of them are different. Let $f = \left(\prod_{i=1}^{\ell} f_i\right) \left(\prod_{1\leq i<j\leq \ell} (f_i - f_j)\right) \in k[[Q]]$. By Lemma 2.5 and Lemma 3.14 one sees that $RW(B, Q)$ implies $RS(B, Q)$.

Corollary 3.15. *Let (B, Q) be a framed space germ over k. Two statements for (B, Q), $RS(B, Q)$ and $RW(B, Q)$ are equivalent.*

Note that if $\sharp Q = \dim B$, then the first framed resolution game above with $\ell = 1$ is regarded as the resolution game in INTRODUCTION with additional procedures. It follows from Lemma 3.11.2. Thus, if Q is a parameter system of B and if $RW(B, Q)$ is true, then any non-zero element of B can be resolved.

We hope to show that for any framed space germ (B, Q) the statement $RW(B, Q)$ is true.

Note that if $\sharp Q = \dim k[[Q]] = 1$, then every non-zero element in $k[[Q]]$ has a normal crossing with respect to Q.

Lemma 3.16. *If the basic dimension $\sharp Q$ of a framed space germ (B, Q) is equal to one, then $RW(B, Q)$ is true.*

We hope to apply induction on the basic dimension $\sharp Q$ to show the statement $RW(B, Q)$.

4. FRAMED RESOLUTION GAMES

In subsequent sections we consider various framed resolution games similar to the first framed resolution game after Lemma 3.13. They have the same principles.

When we start the game, a framed space germ (B, P) over k and finite numbered elements $f_1, f_2, \dots , f_\ell \in B$ called the *data* of the game are given. The data have to satisfy certain conditions called the *initial conditions* of the game. It has two players $\langle I\rangle$ and $\langle II\rangle$, and they repeat steps under the same rule. At each step, if all of certain statements for $(B, P, f_1, f_2, \dots , f_\ell)$ called the *winning rules* are true, then the player $\langle I\rangle$ wins and they terminate the game. Otherwise, $\langle I\rangle$ has two choices. In the first choice, $\langle I\rangle$ chooses a subset $C \subset Q$ with $\sharp C \geq 2$. Then, the player $\langle II\rangle$ chooses an element $x_0 \in C$ and a map $\lambda : C - \{x_0\} \to k$. Let (B', P') be the elementary monoidal transform of (B, P) associated with the datum (C, x_0, λ), and by f'_i for $1 \leq i \leq \ell$ we denote the image of f_i by the monoidal transformation $B \to B'$. After replacing the pair $((B, P), (f_1, f_2, \dots , f_\ell))$ by the pair $((B', P'), (f'_1, f'_2, \dots , f'_\ell))$, they proceed to the next step. In the second choice, $\langle I\rangle$ chooses a parameter system P_1 of $k[[P]]$. After replacing the pair $((B, P), (f_1, f_2, \dots , f_\ell))$ by the pair $((B, P_1), (f_1, f_2, \dots , f_\ell))$, they proceed to the next step.

The game is characterized by the number ℓ of the data, the initial conditions and the winning rules.

Example 4.1. At the first framed resolution game, we have

- The number of the data: Any positive integer.
- The initial conditions: For every $1 \leq i \leq \ell$ $f_i \neq 0$ and $f_i \in k[[P]]$.
- The winning rules:
  1. The product $f_1 f_2 \cdots f_\ell \in k[[P]]$ has a normal crossing with respect to P.
  2. The ideal in $k[[P]]$ generated by any non-empty subset of $\{f_1, f_2, \dots , f_\ell\}$ is a principal ideal.

Some are called a framed resolution game of *restricted type*, and they have slightly different (but essentially same) principles. In the game of restricted type the player $\langle I\rangle$ cannot exchange the frame. Replacing the latter half of the above principle of the framed resolution game after "Otherwise," by the following sentences, we obtain the principle of the framed resolution game of restricted type:

Otherwise, $\langle I\rangle$ chooses a subset $C \subset P$ with $\sharp C \geq 2$. The player $\langle II\rangle$ chooses an element $x_0 \in C$ and a map $\lambda : C - \{x_0\} \to k$. Let (B', P') be the elementary monoidal transform of (B, P) associated with the datum (C, x_0, λ), and by f'_i for $1 \leq i \leq \ell$ we denote the image of f_i by the monoidal transformation $B \to B'$. After replacing the pair $((B, P), (f_1, f_2, \dots , f_\ell))$ by the pair $((B', P'), (f'_1, f'_2, \dots , f'_\ell))$, they proceed to the next step.

Also the game of restricted type is characterized by the number ℓ of the data, the initial conditions and the winning rules.

5. WEIERSTRASS REPRESENTATION

The Weierstrass preparation theorem (Corollary 2.11) provides us a method for induction on the basic dimension $\sharp Q$.

Definition 5.1. Let (B, Q) be a framed space germ, and $A = k[[Q]]$. A triplet (h, g, t) satisfying the following four conditions is called a *Weierstrass representation under* (B, Q).

1. $h \in A$, $g \in A$ and $t \in Q$.
   We write $Q = \{t, x_1, x_2, \dots, x_n\}$, and $Z = k[[Q - \{t\}]]$ below. Note that $A = k[[t, x_1, x_2, \dots, x_n]]$ and $Z = k[[x_1, x_2, \dots, x_n]]$.
2. There exist non-negative integers $s_1, s_2, \dots, s_n$ with
   $$\frac{h}{x_1^{s_1} x_2^{s_2} \cdots x_n^{s_n}} \in A^*.$$
3. The pair (g, t) is a semi-Weierstrass pair of the ring A under the parameter system Q. In other words, there are a non-negative integer m and m elements $G_1, G_2, \dots, G_m \in Z$ with
   $$g = t^m + \sum_{j=1}^{m} G_j t^{m-j}.$$
4. For every integer j with $1 \leq j \leq m$
   $$\operatorname{ord}(G_j) \geq j.$$

If (h, g, t) satisfies only conditions 1–3, then we call (h, g, t) a *semi-Weierstrass representation.* The frame Q of the space germ B is called the *upper frame* of (h, g, t), while the frame $P = Q - \{t\}$ of the space germ B is called the *lower frame* of (h, g, t). The framed space germ (B, Q) is called the *upper framed space germ* of (h, g, t), and (B, P) is called the *lower framed space germ* of (h, g, t).

We say also that (h, g, t) is a semi-Weierstrass representation *over* (B, P), referring to the lower framed space germ (B, P), not to the upper framed space germ (B, Q). If (h, g, t) is one *over* (B, P), then (B, P) is the lower framed space germ of (h, g, t), $t \notin P$, and the union $P \cup \{t\}$ is a partial parameter system of B. If it is one *under* (B, Q), then (B, Q) is the upper framed space germ of (h, g, t) and $t \in Q$.

Let $f \in A$ be a non-zero element. If a semi-Weierstrass representation (h, g, t) satisfies $f = hg$, then we say that (h, g, t) is a semi-Weierstrass representation *of* f.

If a semi-Weierstrass pair (g, t) of A under Q has a certain property X, then, for simplicity, we say that the semi-Weierstrass representation (h, g, t) under (B, Q) has a property X.

Lemma 5.2. *Let (B, Q) be a framed space germ, and $f \in k[[Q]]$ a non-zero element.*

1. *There exists a coordinate transform (B, Q') of (B, Q) such that there exists a separable Weierstrass representation (h, g, t) of f under (B, Q') satisfying $h \in k[[Q]]^*$.*
2. *Any semi-Weierstrass representation (h, g, t) of f satisfies* $\operatorname{ord}(f) = \operatorname{ord}(g) + \operatorname{ord}(h)$, $\operatorname{sord}(h) = 0$, $\operatorname{sord}(f) = \operatorname{sord}(g)$, *and* $\operatorname{nord}(f) = \operatorname{nord}(g) + \operatorname{nord}(h)$.
3. *Any Weierstrass representation (h, g, t) of f with $h \in k[[Q]]^*$ satisfies* $\operatorname{ord}(f) = \operatorname{ord}(g) = \deg(g, t)$, $\operatorname{sord}(f) = \operatorname{sord}(g)$, *and* $\operatorname{nord}(f) = \operatorname{nord}(g)$.

Proof. 1. Note that $k[[Q']] = k[[Q]]$ by definition of a coordinate transform. It follows from Lemma 2.20.
2. See Definition 2.8 and Lemma 2.9 for definition of sord and nord.
3. It follows from 2 and Lemma 2.17.4. □

Lemma 5.3. *Let (h, g, t) be a semi-Weierstrass representation over a framed space germ (B, P). Let $\varphi : (B, P) \to (B', P')$ be a morphism with a normal crossing from the lower framed space germ. We write $h' = \varphi(h)$, $g' = \varphi(g)$, and $t' = \varphi(t)$.*

1. *$Q' = P' \cup \{t'\}$ is a frame of B'.*
2. *Triplet (h', g', t') is a semi-Weierstrass representation over (B', P') and under (B', Q').*
3. $\mathrm{w}(g,\, t) \le \mathrm{w}(g',\, t')$, $\mathrm{ord}(g) \le \mathrm{ord}(g')$, $\deg(g,\, t) = \deg(g',\, t')$.
4. *If $(g,\, t)$ is separable, then also $(g',\, t')$ is separable.*
5. *Assume moreover that (h, g, t) is a Weierstrass representation. Also $(h'$, $g', t')$ is a Weierstrass representation.* $\mathrm{ord}(g) = \mathrm{ord}(g')$, $\mathrm{nord}(g) \le \mathrm{nord}(g')$, $\mathrm{sord}(g) \ge \mathrm{sord}(g')$.

Proof. 1. It follows from Lemma 3.6.
2. It is easy to check the first and the third condition in Definition 5.1. Since φ preserves normal crossings, also the second condition is satisfied.
3. For every $f \in k[[P \cup \{t\}]]$ we have $\mathrm{ord}(f) \le \mathrm{ord}(\varphi(f))$.
4. Easy.
5. By Lemma 2.17.4 we have $\deg(g,\, t) = \mathrm{ord}(g)$. By 3 and Lemma 2.17.2 we have $\mathrm{ord}(g) \le \mathrm{ord}(g') \le \deg(g', t') = \deg(g,\, t) = \mathrm{ord}(g)$. We have $\deg(g', t') = \mathrm{ord}(g')$, and again by Lemma 2.17.4 one concludes that (h', g', t') is a Weierstrass representation. The equality $\mathrm{ord}(g) = \mathrm{ord}(g')$ follows from the above. Let D be the field of fractions of $k[[P]]$, and D' the field of fractions of $k[[P']]$. Let $g = g_1^{\beta_1} g_2^{\beta_2} \cdots g_r^{\beta_r}$ be the irreducible decomposition of g in $D[t]$. We assume that the coefficient of the term with the highest degree in $g_\nu \in D[t]$ is equal to 1 for every $1 \le \nu \le r$, and that for $1 \le \nu \le s$ $\deg(g_\nu, t) = 1$ and for $s < \nu \le r$ $\deg(g_\nu, t) > 1$. Since $\deg(g,\, t) = \mathrm{ord}(g)$, by Lemma 2.18.3 we have $\deg(g_\nu, t) = \mathrm{ord}(g_\nu)$ for every $1 \le \nu \le r$, and $g = g_1^{\beta_1} g_2^{\beta_2} \cdots g_r^{\beta_r}$ gives the irreducible decomposition in $k[[P \cup \{t\}]]$. Thus $\mathrm{nord}(g) = \sum_{\nu=1}^{s} \beta_\nu$. On the other hand, we have a decomposition $g' = \varphi(g_1)^{\beta_1} \varphi(g_2)^{\beta_2} \cdots \varphi(g_r)^{\beta_r}$ in $D'[t']$. Since $\deg(g', t') = \mathrm{ord}(g')$, by Lemma 2.18.1 we have $\deg(\varphi(g_\nu), t') = \mathrm{ord}(\varphi(g_\nu))$ for every $1 \le \nu \le r$. Thus $\mathrm{ord}(\varphi(g_\nu)) = \deg(\varphi(g_\nu), t') = \deg(g_\nu, t) = 1$ for $1 \le \nu \le s$. One concludes $\mathrm{nord}(g') \ge \sum_{\nu=1}^{s} \beta_\nu = \mathrm{nord}(g)$.

The last equality follows from Lemma 2.9.2. □

The semi-Weierstrass representation (h', g', t') is called the *pull-back* of (h, g, t) by φ. If (h, g, t) is a semi-Weierstrass (resp. Weierstrass) representation of $f \in k[[P \cup \{t\}]]$, then the pull-back (h', g', t') is a semi-Weierstrass (resp. Weierstrass) representation of $\varphi(f) \in k[[P' \cup \{t'\}]]$.

We denote the integral part of a real number r by $[r]$ (The Gauss symbol), and the part under the decimal point by $\langle r \rangle$.

$$\begin{aligned} [r] &= \max\{i \in \mathbb{Z} \mid i \le r\}, \\ \langle r \rangle &= r - [r]. \end{aligned}$$

Definition 5.4. Let (h, g, t) be a semi-Weierstrass representation over a space germ (B, P). We write $P = \{x_1, x_2, \ldots, x_n\}$, $Z = k[[P]]$. Let $m = \deg(g, t)$ and $G_1, G_2, \ldots, G_m \in Z$ be the coefficient functions of (g, t).

$$g = t^m + \sum_{j=1}^{m} G_j t^{m-j}.$$

1. We say that a semi-Weierstrass pair (g,t) satisfies the *Abhyankar condition*, if the following condition is satisfied: If $\mathrm{w}(g,\,t)<\infty$, then there exist n non-negative integers $b_1,b_2,\dots,b_n$ such that
   (a)
   $$\frac{G_j}{(x_1^{b_1}x_2^{b_2}\cdots x_n^{b_n})^j}\in Z$$
   for every $1\le j\le m=\deg(g,\,t)\ge 1$, and
   (b)
   $$\min\{\frac{1}{j}\operatorname{ord}(\frac{G_j}{(x_1^{b_1}x_2^{b_2}\cdots x_n^{b_n})^j})\mid 1\le j\le m\}<1.$$
   The non-negative integers $b_1,b_2,\dots,b_n$ above are called the *characteristic numbers* of (g,t).
2. We say that (g,t) satisfies the *strong Abhyankar condition*, if the following three conditions are simultaneously satisfied:
   (c) Let $J=\{j\in\mathbb{Z}\mid 1\le j\le m, G_j\neq 0\}$. For every $j\in J$ there exist n integers $b_{j1},b_{j2},\dots,b_{jn}$ with
   $$\frac{G_j}{x_1^{b_{j1}}x_2^{b_{j2}}\cdots x_n^{b_{jn}}}\in Z^*.$$
   (d) The subset
   $$E=\{e\in J\mid \frac{G_j^{m!/j}}{G_e^{m!/e}}\in Z \text{ for every } j\in J\}$$
   of J is not empty.
   (e) There exists an element $e\in E$ with
   $$\sum_{i=1}^{n}\langle\frac{b_{ei}}{e}\rangle<1.$$

Remark . The strong Abhyankar condition is due to Abhyankar. (Abhyankar [2], p.285–p.299, Appendix.)

Note that if (g,t) satisfies the strong Abhyankar condition, then $\mathrm{w}(g,\,t)<\infty$, while any (g,t) with $\mathrm{w}(g,\,t)=\infty$ satisfies the Abhyankar condition by definition.

Lemma 5.5. *We consider conditions in Definition 5.4.*

1. *If (g,t) satisfies the Abhyankar condition, and $\mathrm{w}(g,\,t)<\infty$, then characteristic numbers $b_1,b_2,\dots,b_n$ are uniquely defined depending on (g,t). In other words, n numbered non-negative integers $b_1,b_2,\dots,b_n$ satisfying* (b) *are unique, if they exist.*
2. *If (g,t) satisfies the Abhyankar condition, $\mathrm{w}(g,\,t)<\infty$, and (g,t) is a Weierstrass pair, then at least one of characteristic numbers $b_1,b_2,\dots,b_n$ is not zero.*
3. *Non-negative integers b_{ji}, $j\in J$, $1\le i\le n$ in* (c) *are uniquely determined depending on (g,t), if they exist. Therefore, if* (c) *is satisfied, then* (e) *has definite meaning.*
4. *If* (c) *and* (d) *are satisfied, then for a fixed number i with $1\le i\le n$, the rational number*
   $$\frac{b_{ei}}{e}$$

does not depend on the choice of an element $e \in E$. Therefore, the inequality in (e) *holds for every element $e \in E$.*

Proposition 5.6. *Let (g, t) be a semi-Weierstrass pair of a ring $k[[x]]$ over a partial parameter space P. If (g, t) satisfies the strong Abhyankar condition, then (g, t) satisfies the Abhyankar condition. In other words, the strong Abhyankar condition implies the Abhyankar condition.*

Theorem 5.7 (Abhyankar [2], p.285–p.299, Appendix.)**.** *Let (B, P) be a framed space germ over k.*

1. *We consider the* second framed resolution game *characterized by the conditions below:*
   - *The number of the data:* $\sharp\, P + 3$.
   - *The initial conditions: Let $(h, g, t, \xi_1, \xi_2, \dots, \xi_n)$ denote the data.*
     (a) *(h, g, t) is a Weierstrass representation over (B, P) with* $\mathrm{w}(g,\, t) < \infty$.
     (b) $P = \{\xi_1, \xi_2, \dots, \xi_n\}$.
   - *The winning rules:*
     (a) *The triplet (h, g, t) is a Weierstrass representation over (B, P) satisfying conditions* (c) *and* (d) *in Definition 5.4*
     (b) *The product $\xi_1 \xi_2 \cdots \xi_n$ has a normal crossing with respect to P.*

   Assume that $RW(B, P)$ is true. Then, the player $\langle I \rangle$ can always win this game.
2. *We consider the* third framed resolution game. *The third game is of restricted type and is characterized by the conditions below:*
   - *The number of the data:* 3.
   - *The initial conditions: The data (h, g, t) is a Weierstrass representation over (B, P) satisfying* (c) *and* (d) *in Definition 5.4.*
   - *The winning rules: The triplet (h, g, t) is a Weierstrass representation over (B, P) satisfying the strong Abhyankar condition.*

   The player $\langle I \rangle$ can always win this game.

Proof. Let $m = \deg(g,\, t)$, $G_1, G_2, \dots, G_m \in k[[P]]$ be the coefficient functions of $(g,\, t)$, and $J = \{j \in \mathbb{Z} \mid 1 \leq j \leq m, G_j \neq 0\}$. Since $\mathrm{w}(g,\, t) < \infty$, we have $J \neq \emptyset$.
1. Note that at the beginning of each step of the game, the triplet (h, g, t) is a Weierstrass representation over (B, P), if the product $\xi_1 \xi_2 \cdots \xi_n$ has a normal crossing with respect to P. Since $RW(B, P)$ is true, also $RS(B, P)$ is true by Corollary 3.15. Consider finite elements $G_j^{m!/j}$, $j \in J$ and $x_1, x_2, \dots, x_n$. Claim 1 follows from $RS(B, P)$.
2. It follows from Lemma 5.8 below. ☐

Lemma 5.8 (Abhyankar [2], p.290, Reduction Lemma)**.** *Let (B, P) be a framed space germ over k. We consider the* fourth framed resolution game. *The fourth game is of restricted type and is characterized by the conditions below: A given positive integer is denoted by e.*

- *The number of the data:* 1.
- *The initial conditions: The data G is a non-zero elements in $k[[P]]$ with a normal crossing with respect to $P = \{y_1, y_2, \dots, y_n\}$.*
- *The winning rules: The non-negative integers $b_1, b_2, \dots, b_n$ defined by the condition*

$$\frac{G}{y_1^{b_1} y_2^{b_2} \cdots y_n^{b_n}} \in k[[P]]^*$$

satisfy

$$\sum_{i=1}^{n} \langle \frac{b_i}{e} \rangle < 1.$$

The player $\langle I \rangle$ can always win this game.

Remark . Note that the above game is of restricted type. Thus the framed space germ (B', P') obtained after several steps of the game is an IEMT of the originally given (B, P), and the induced homomorphism $(B, P) \to (B', P')$ has a normal crossing (Lemma 3.13.3).

Assume, moreover, that for given a finite number of elements $G_1, G_2, \dots, G_\ell \in k[[P]]$ and positive integers $\alpha, \alpha_1, \alpha_2, \dots, \alpha_m$ the product $GG_1G_2 \cdots G_\ell$ has a normal crossing with respect to P and the ideal in $k[[P]]$ generated by $G^\alpha, G_1^{\alpha_1}, G_2^{\alpha_2}, \dots, G_\ell^{\alpha_\ell}$ is a principal ideal generated by G^α.

Let $G_i' \in k[[P']]$ we denote the image of $G_i \in k[[P]]$ by $k[[P]] \to k[[P']]$. Then, the product $G'G_1'G_2' \cdots G_\ell'$ has a normal crossing with respect to P' and the ideal in $k[[P']]$ generated by $(G')^\alpha, (G_1')^{\alpha_1}, (G_2')^{\alpha_2}, \dots, (G_m')^{\alpha_m}$ is a principal ideal generated by $(G')^\alpha$.

We can check that Abhyankar's proof is valid under our formulation with framed resolution games.

Corollary 5.9. *Consider the same situation as in Theorem 5.7. We assume that $RW(B, P)$ is true.*

We consider the fifth framed resolution game *characterized by the conditions below:*

- *The number of the data: $\sharp P + 3$.*
- *The initial conditions: Let $(h, g, t, \xi_1, \xi_2, \dots, \xi_n)$ denote the data.*
  1. *(h, g, t) is a Weierstrass representation over (B, P) with* $\mathrm{w}(g,\, t) < \infty$.
  2. *$P = \{\xi_1, \xi_2, \dots, \xi_n\}$.*
- *The winning rules:*
  1. *The triplet (h, g, t) is a Weierstrass representation over (B, P) satisfying the strong Abhyankar condition.*
  2. *The product $\xi_1\xi_2 \cdots \xi_n$ has a normal crossing with respect to P.*

The player $\langle I \rangle$ can always win this game.

We introduce another condition "reductive" for semi-Weierstrass pairs, which is essential in the theory of positive characteristic.

Lemma 5.10. *Let (h, g, t) be a semi-Weierstrass representation over a framed space germ (B, P), $m = \deg(g,\, t)$, and $G_1, G_2, \dots,\, G_m \in k[[P]]$ the coefficient functions of the semi-Weierstrass pair $(g,\, t)$.*

1. *For every element $H \in k[[P]]$ the triplet $(h, g, t+H)$ (resp. the pair $(g,\, t+H)$) is also a semi-Weierstrass representation over the framed space germ (B, P) (resp. a semi-Weierstrass pair of $k[[P \cup \{t\}]]$ over P).* $\deg(g,\, t) = \deg(g,\, t + H)$.

2. *Let* $H \in k[[P]]$*, and* $G'_1, G'_2, \dots, G'_m \in k[[P]]$ *be the coefficient functions of the semi-Weierstrass pair* $(g,\, t+H)$*. Let* $G_0 = G'_0 = 1$*. We have*

$$\begin{aligned} G'_j &= \sum_{i=0}^{j} \binom{m-i}{m-j} (-H)^{j-i} G_i, \\ G_j &= \sum_{i=0}^{j} \binom{m-i}{m-j} H^{j-i} G'_i, \end{aligned}$$

for every $1 \leq j \leq m$*.*

Definition 5.11. Let (B, P) be a framed space germ over k.

1. Let $t \in B$ be an element such that $t \notin P$ and $P \cup \{t\}$ is a partial parameter system of B. The coordinate transform $(B, P \cup \{t+H\})$ of $(B, P \cup \{t\})$ defined for any element $H \in k[[P]]$ is called the *parallel coordinate transform* of $(B, P \cup \{t\})$ along H.
2. Let (h, g, t) be a semi-Weierstrass representation over (B, P), and $H \in k[[P]]$. The semi-Weierstrass representation $(h, g, t+H)$ (resp. the semi-Weierstrass pair $(g,\, t+H)$) is called the *parallel coordinate transform* of (h, g, t) (resp. $(g,\, t)$) along H.
3. An element $H \in k[[P]]$ with $\mathrm{w}(g,\, t) < \mathrm{w}(g,\, t+H)$ is called a *reducing* element of (h, g, t) or of $(g,\, t)$.
4. If (h, g, t) or $(g,\, t)$ has a reducing element $H \in k[[P]]$, then we say that (h, g, t) or $(g,\, t)$ is *reductive.*

Lemma 5.12. *Let* n *be a positive integer and* A *a ring isomorphic to the formal power series ring over* k *with* $n+1$ *variables. Let* $P = \{x_1, x_2, \dots, x_n\}$ *be a partial parameter system of* A *with* $\sharp P = n$*, and* $(g,\, t)$ *a semi-Weierstrass pair of* A *over* P*. Let* $m = \deg(g,\, t)$ *and* $G_1, G_2, \dots, G_m \in k[[P]]$ *the coefficient functions of* $(g,\, t)$*.*

1. *If* (g, t) *is separable, then also* $(g, t+H)$ *is separable for any* $H \in k[[P]]$*.*
2. *For any reducing element* $H \in k[[P]]$ *of* $(g,\, t)$*, we have* $\mathrm{ord}(H) = \mathrm{w}(g,\, t) < \infty$*.*
3. *For an element* $H \in k[[P]]$ *the following two conditions are equivalent:*
   - (a) H *is a reducing element of* $(g,\, t)$*.*
   - (b) $m \geq 1$*,* $H \neq 0$ *and* $\mathrm{in}(g,\, t) = (t + \mathrm{in}(H))^m$*.*
4. *The following four conditions are equivalent:*
   - (a) *Pair* $(g,\, t)$ *is reductive.*
   - (b) *Pair* $(g,\, t)$ *is reductive,* $\mathrm{w}(g,\, t) < \infty$ *and* $m \geq 1$*.*
   - (c) $m \geq 1$*, and there exists an element* $H_0 \in k[[P]]$ *such that* $H_0 \neq 0$ *and* $\mathrm{in}(g,\, t) = (t + H_0)^m$*.*
   - (d) *There exists a unique element* $H_0 \in k[[P]]$ *satisfying* $\mathrm{in}(g,\, t) = (t + H_0)^m$*, and* H_0 *is a non-zero homogeneous polynomial.*
5. *One and only one of following two claims holds:*
   - (a) *Pair* $(g,\, t)$ *is not reductive.*
   - (b) *Pair* $(g,\, t)$ *is reductive, and there exists an element* $H \in k[[P]]$ *such that* $(g,\, t+H)$ *is not reductive.*

Below we assume that $(g,\, t)$ *is reductive. Let* $m = lq$ *be the* p*-decomposition of* m*.* (*See the latter half of* INTRODUCTION *for the definition.*)

6. *Weight* $\mathrm{w}(g,\, t)$ *is an integer.*
7. *For an element* $H \in k[[P]]$ *the following two conditions are equivalent:*

(a) *H is a reducing element of (g, t).*
(b) *$H \neq 0$ and $\mathrm{in}(G_q) = l\,\mathrm{in}(H)^q$.*

Proof. Note Lemma 5.10.2 and Lemma 2.2. For every integer j with $1 \leq j \leq m$ and $j \not\equiv 0 \mod q$ we have $\binom{m}{j} = 0$. For an integer j with $1 \leq j \leq m$ and $j \equiv 0 \mod q$ we have $\binom{m}{j} = \binom{lq}{iq} = \binom{l}{i}$, where $i = j/q$. In particular, $\binom{m}{q} = \binom{l}{1} = l \neq 0$.

By Lemma 2.1, we obtain implication 3.(b)⇒(a). □

We here extend the Abhyankar condition for m-tuples of functions.

Definition 5.13. Let (B, P) be a framed space germ over k, m a non-negative integer, and $G_1, G_2, \ldots, G_m \in Z = k[[P]]$ m numbered elements. We write $P = \{x_1, x_2, \ldots, x_n\}$.

1. If $m = 0$, we define $\mathrm{w}() = \infty$. If $m > 0$, we define $\mathrm{w}(G_1, G_2, \ldots, G_m) = \min\{\mathrm{ord}(G_j)/j \mid 1 \leq j \leq m\}$. We call $\mathrm{w}(G_1, G_2, \ldots, G_m) \in \mathbb{Q} \cup \{\infty\}$ the *weight* of $(G_1, G_2, \ldots, G_m)$.
2. We say that a m-tuple $(G_1, G_2, \ldots, G_m)$ satisfies the *Abhyankar condition*, if the following condition is satisfied: If $\mathrm{w}(G_1, G_2, \ldots, G_m) < \infty$, then there exist n non-negative integers $b_1, b_2, \ldots, b_n$ such that
   (a)
   $$\frac{G_j}{(x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n})^j} \in Z$$
   for every $1 \leq j \leq m$, and
   (b)
   $$\min\{\frac{1}{j}\,\mathrm{ord}(\frac{G_j}{(x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n})^j}) \mid 1 \leq j \leq m\} < 1.$$
   The non-negative integers $b_1, b_2, \ldots, b_n$ above are called the *characteristic numbers* of $(G_1, G_2, \ldots, G_m)$.

Lemma 5.14. 1. *If $(G_1, G_2, \ldots, G_m)$ satisfies the Abhyankar condition, then characteristic numbers $b_1, b_2, \ldots, b_n$ are uniquely defined depending on $(G_1, G_2, \ldots, G_m)$.*
2. *Let $t \in B$ be an element such that $t \notin P$ and $P \cup \{t\}$ is a partial parameter system of B. Let $g = t^m + \sum_{j=1}^m G_j t^{m-j}$.*
   (a) $\mathrm{w}(G_1, G_2, \ldots, G_m) = \mathrm{w}(g, t)$.
   (b) *The m-tuple $(G_1, G_2, \ldots, G_m)$ satisfies the Abhyankar condition, if and only if, the semi-Weierstrass pair (g, t) satisfies the Abhyankar condition.*
   (c) *If the equivalent conditions in* (b) *are satisfied, then the characteristic numbers of $(G_1, G_2, \ldots, G_m)$ and the characteristic numbers of (g, t) coincide.*
3. *Let $v \in Z^*$.*
   (a) $\mathrm{w}(G_1, G_2, \ldots, G_m) = \mathrm{w}(vG_1, v^2G_2, \ldots, v^mG_m)$.
   (b) *An m-tuple $(G_1, G_2, \ldots, G_m)$ satisfies the Abhyankar condition, if and only if, $(vG_1, v^2G_2, \ldots, v^mG_m)$ satisfies the Abhyankar condition.*
   (c) *If the equivalent conditions in* (b) *are satisfied, then the characteristic numbers of $(G_1, G_2, \ldots, G_m)$ and the characteristic numbers of $(vG_1, v^2G_2, \ldots, v^mG_m)$ coincide.*

Proof. 1. See Lemma 5.5.1.
2. See Definition 5.4.1. ☐

Lemma 5.15. *Let (B, P) be a framed space germ over k, and (h, g, t) a Weierstrass representation over (B, P). Assume that (h, g, t) satisfies the Abhyankar condition,* $\mathrm{w}(g,\, t) < \infty$, *and (h, g, t) is reductive. We denote $P = \{x_1, x_2, \dots, x_n\}$, $Z = k[[P]] = k[[x_1, x_2, \dots, x_n]]$, $m = \deg(g, t)$. Let $G_1, G_2, \dots, G_m \in Z$ be the coefficient functions of (g, t), and $b_1, b_2, \dots, b_n$ the characteristic numbers of (g, t). Note that $m \geq 1$ by assumption. Let $m = lq$ be the p-decomposition of m.*

The following claims 1–9 hold:

1. $\mathrm{w}(g,\, t) = \sum_{i=1}^{n} b_i$.
2. *At least one of the characteristic numbers $b_1, b_2, \dots, b_n$ is not zero.*
3. *There exists a unique element u_j satisfying*
$$G_j = u_j (x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n})^j, \quad u_j \in Z$$
for every $1 \leq j \leq m$.
4. *There exists a unique element $a \in k$ satisfying*
$$\mathrm{in}(g,\, t) = (t + a x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n})^m, \quad a \neq 0.$$
5. $u_j(0) = \dbinom{m}{j} a^j$ *for every $1 \leq j \leq m$, where we denote* $u_j(0) = u_j \bmod M(Z) \in Z/M(Z) \cong k$. *Besides, $u_q \in Z^*$, and $u_m \in Z^*$.*
6. *The following three conditions are equivalent for $H \in Z$:*
   (a) *H is a reducing element of (g, t). In other words,* $\mathrm{w}(g,\, t) < \mathrm{w}(g,\, t + H)$.
   (b) $\mathrm{in}(G_q) = l\, \mathrm{in}(H)^q$.
   (c) $\mathrm{in}(H) = a x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}$.
7. *There exists a reducing element $H \in Z$ of $(g,\, t)$ satisfying*
$$\frac{H}{x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}} \in Z^*.$$
8. *Let $H \in Z$ be a reducing element of $(g,\, t)$ satisfying the condition in* 7. *Then, there exist m numbered elements $F_1, F_2, \dots, F_m \in M(Z)$ satisfying*
$$g = (t + H)^m + \sum_{i=1}^{m} (x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n})^j F_j (t + H)^{m-j}.$$
9. *Assume that non-negative integers $c_1, c_2, \dots, c_n$ and elements $F'_1, F'_2, \dots, F'_m \in M(Z)$ satisfy*
$$G_j = (x_1^{c_1} x_2^{c_2} \cdots x_n^{c_n})^j F'_j$$
for every $1 \leq j \leq m$. Then, $(F'_1, F'_2, \dots, F'_m)$ satisfies the Abhyankar condition. Let $d_1, d_2, \dots, d_n$ be the characteristic numbers of $(F'_1, F'_2, \dots, F'_m)$. The following three claims hold:
   (a) *$b_i = c_i + d_i$ for every $1 \leq i \leq n$.*
   (b) $\mathrm{w}(F'_1, F'_2, \dots, F'_m) = \sum_{i=1}^{n} d_i$.
   (c) *At least one of $d_1, d_2, \dots, d_n$ is not zero.*

Proof. 1. By Lemma 5.12.6 $\mathrm{w}(g,\, t)$ is an integer. By the Abhyankar condition for $(g,\, t)$ we have
$$0 \leq \mathrm{w}(g,\, t) - \sum_{i=1}^{n} b_i = \min\{\frac{1}{j}\, \mathrm{ord}(\frac{G_j}{(x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n})^j})\} < 1.$$

2. It follows from 1 and Lemma 2.17.4.
3. It follows from that $(G_1, G_2, \dots, G_m)$ satisfies the Abhyankar condition.
4 and 5. By 1 $\min\{\mathrm{ord}(u_j)/j \mid 1 \le j \le m\} = \min\{\mathrm{ord}(G_j/(x_1^{b_1}x_2^{b_2}\cdots x_n^{b_n})^j)/j \mid 1 \le j \le m\} = 0$. Thus we have $\mathrm{in}(g,\, t) = t^m + \sum_{i=1}^m u_j(0)(x_1^{b_1}x_2^{b_2}\cdots x_n^{b_n})^j t^{m-j}$. On the other hand, by Lemma 5.12.4 there is a unique element $K \in Z$ with $\mathrm{in}(g,\, t) = (t + K)^m$, and $K \neq 0$. Thus $u_j(0)(x_1^{b_1}x_2^{b_2}\cdots x_n^{b_n})^j = \binom{m}{j}K^j$ for every $1 \le j \le m$. By Lemma 2.2 $\binom{m}{q} = \binom{l}{1} = l \neq 0$. Thus we have $u_q(0)(x_1^{b_1}x_2^{b_2}\cdots x_n^{b_n})^q = l\,K^q \neq 0$. Let $a \in k$ be an element satisfying $u_q(0) = la^q$. We have $a \neq 0$, since $u_q(0) \neq 0$. We have $K = ax_1^{b_1}x_2^{b_2}\cdots x_n^{b_n}$, since $l(K - ax_1^{b_1}x_2^{b_2}\cdots x_n^{b_n})^q = lK^q - l(ax_1^{b_1}x_2^{b_2}\cdots x_n^{b_n})^q = 0$. The uniqueness of a follows from the uniqueness of K. Also we obtain the former half of 5. The latter half of 5 is easy.
6. It follows from Lemma 5.12.7 and above 4.
7. By 6 one knows that $H = ax_1^{b_1}x_2^{b_2}\cdots x_n^{b_n}$ satisfies the condition.
8. Assume that $H \in Z$, $h = H/(x_1^{b_1}x_2^{b_2}\cdots x_n^{b_n}) \in Z^*$ and $\mathrm{w}(g,\, t) < \mathrm{w}(g,\, t + H)$. We have $h(0)x_1^{b_1}x_2^{b_2}\cdots x_n^{b_n} = \mathrm{in}(H) = ax_1^{b_1}x_2^{b_2}\cdots x_n^{b_n}$ by 6. Thus $h(0) = a$.

Now, we can write

$$g = t^m + \sum_{j=1}^m u_j(x_1^{b_1}x_2^{b_2}\cdots x_n^{b_n})^j t^{m-j} = (t+H)^m + \sum_{j=1}^m \Phi_j(t+H)^{m-j}$$

with $\Phi_1, \Phi_2, \dots, \Phi_m \in Z$. Let $u_0 = 1$. By Lemma 5.10.2 we have

$$\begin{aligned}\Phi_j &= \sum_{i=0}^j \binom{m-i}{m-j}(-H)^{j-i}u_i(x_1^{b_1}x_2^{b_2}\cdots x_n^{b_n})^i \\ &= (x_1^{b_1}x_2^{b_2}\cdots x_n^{b_n})^j \sum_{i=0}^j \binom{m-i}{m-j}(-h)^{j-i}u_i\end{aligned}$$

for every $1 \le j \le m$. Putting $F_j = \sum_{i=0}^j \binom{m-i}{m-j}(-h)^{j-i}u_i \in Z$, we have $\Phi_j = (x_1^{b_1}x_2^{b_2}\cdots x_n^{b_n})^j F_j$ for $1 \le j \le m$. For the constant term $F_j(0)$ of F_j we have

$$\begin{aligned}F_j(0) &= \sum_{i=0}^j \binom{m-i}{m-j}(-h(0))^{j-i}u_i(0) = \sum_{i=0}^j \binom{m-i}{m-j}(-a)^{j-i}\binom{m}{i}a^i \\ &= a^j \sum_{i=0}^j \binom{m-i}{j-i}\binom{m}{i}(-1)^{j-i} = a^j \sum_{i=0}^j \binom{j}{i}\binom{m}{j}(-1)^{j-i} \\ &= a^j\binom{m}{j}\{1 + (-1)\}^j = 0.\end{aligned}$$

We used 5 and Lemma 2.1. Thus $F_j \in M(Z)$.
9. Easy. □

Definition 5.16. Let (h, g, t) be a reductive Weierstrass representation over $(B, \{x_1.x_2, \dots, x_n\})$ satisfying the Abhyankar condition. Let $b_1, b_2, \dots, n_n$ be characteristic numbers of (g, t). A reducing element $H \in Z$ of (g, t) satisfying $H/(x_1^{b_1}x_2^{b_2}\cdots x_n^{b_n}) \in Z^*$ is called an *effective* reducing element of (g, t).

6. The first reduction

Following Abhyankar's ideas in [2], p.285–p.299, Appendix, in this section we add our concept "reductive" to his ideas. We will show that the Abhyankar condition plus the condition "not reductive" implies that the singularity can be resolved even in characteristic positive.

Theorem 6.1 (Abhyankar [2], p.292–p.293, Good Point Lemma.). *Let (B, Q) be a framed space germ, $f \in k[[Q]]$ a non-zero element, and (h, g, t) a Weierstrass representation of f under (B, Q).*

We write $Q - \{t\} = P = \{x_1, x_2, \dots, x_n\}$, and $Z = k[[P]]$. Let $m = \deg(g,\, t)$, and $G_1, G_2, \dots, G_m \in Z$ be the coefficient functions of $(g,\, t)$.

Assume that for every integer j with $1 \le j \le m$ $G_j/x_1^j \in Z$. Let (B', Q') be the elementary monoidal transform of (B, Q) associated with the datum $(\{t, x_1\}, x_1, \lambda)$, where λ is a map $\lambda : \{t\} \to k$. We denote $\lambda(t) \in k$ again by λ for simplicity. We identify elements in B and their images in B' by the elementary transformation, and denote them by the same symbol.

1. *$Q' = \{(t/x_1) - \lambda\} \cup P$.*
2. *The set P is also a frame of B'.*
3. *Let*
$$t' = \frac{t}{x_1} - \lambda,\ g' = \frac{g}{x_1^m},\ h' = hx_1^m.$$
The triplet (h', g', t') is a semi-Weierstrass representation of f over (B', P'). $\deg(g',\, t') = \deg(g,\, t)$.
4. *$\operatorname{ord}(g') \le \operatorname{ord}(g)$. Besides, $\operatorname{ord}(g') = \operatorname{ord}(g)$ if and only if (h', g', t') is a Weierstrass representation of f.*
5. *If (h, g, t) is not reductive and if $\operatorname{ord}(g') = \operatorname{ord}(g) > 0$, then $\lambda = 0$ and (h', g', t') is not reductive, either.*
6. *If $\lambda = 0$ and if (h, g, t) satisfies the Abhyankar condition, then also (h', g', t') satisfies the Abhyankar condition.*
7. *If $\lambda = 0$ and if $\mathrm{w}(g,\, t) < \infty$, then $\mathrm{w}(g',\, t') = \mathrm{w}(g,\, t) - 1$. If $\lambda = 0$ and if $\mathrm{w}(g,\, t) = \infty$, then $\mathrm{w}(g',\, t') = \infty$.*
8. *$\operatorname{sord}(g') \le \operatorname{sord}(g)$.*

Remark . Claim 5 does not appear in Abhyankar [2].

Proof. We have $k[[Q']] = k[[t', x_1, x_2, \dots, x_n]] \supset k[[Q]] = k[[t, x_1, x_2, \dots, x_n]] \supset Z = k[[x_1, x_2, \dots, x_n]]$.

3. Obviously $f = h'g'$. Since $g = t^m + \sum_{j=1}^m G_j t^{m-j}$ and $t = (t' + \lambda)x_1$, we have
$$g' = \frac{g}{x_1^m} = (t' + \lambda)^m + \sum_{j=1}^{m} \frac{G_j}{x_1^j}(t' + \lambda)^{m-j}.$$
By assumption one knows $g' \in k[[Q']]$.

Since (h, g, t) is a Weierstrass representation, we have non-negative integers $s_1, s_2, \dots, s_n$ with
$$\frac{h'}{x_1^{s_1+m} x_2^{s_2} \cdots x_n^{s_n}} = \frac{h}{x_1^{s_1} x_2^{s_2} \cdots x_n^{s_n}} \in k[[Q]]^* \subset k[[Q']]^*.$$

Let
$$G'_j = \binom{m}{m-j}\lambda^m + \sum_{i=1}^{j} \binom{m-i}{m-j} \lambda^{j-i} \frac{G_i}{x_1^i}$$

for $1 \leq j \leq m$. By assumption one has $G'_j \in Z$ for $1 \leq j \leq m$. One can check $g' = (t')^m + \sum_{i=1}^m G'_j (t')^{m-j}$.
4. By 3 and by Lemma 2.17.2 $\mathrm{ord}(g') \leq \deg(g',\, t') = m = \mathrm{ord}(g)$.
5. Since $m = \mathrm{ord}(g') = \mathrm{ord}(g) > 0$, for every $1 \leq j \leq m$ we have $\mathrm{ord}(G'_j) \geq j$, and $G'_j \in M(Z)$.

On the other hand, we have

$$\frac{G_j}{x_1^j} = \binom{m}{m-j}(-\lambda)^j + \sum_{i=1}^{j} \binom{m-i}{m-j}(-\lambda)^{j-i} G'_i.$$

Thus, for every $1 \leq j \leq m$ there is an element $R_j \in M(Z)$ with

$$G_j = \binom{m}{m-j}(-\lambda)^j x_1^j + x_1^j R_j.$$

Assume $\lambda \neq 0$. We have $\mathrm{in}(g,\, t) = (t - \lambda x_1)^m$. By Lemma 5.12.4 $(g,\, t)$ is reductive, which contradicts the assumption.

Assume $\lambda = 0$. For every $1 \leq j \leq m$ we have

$$G'_j = \frac{G_j}{x_1^j}. \tag{*}$$

Let $\varphi(t) = \mathrm{in}(g,\, t)$. By (*) one has $\varphi(x_1 t') = x_1^m \mathrm{in}(g',\, t')$.

Assume that (h', g', t') is reductive. By Lemma 5.12.4 there is an element $H \in Z$ with $H \neq 0$ and $\mathrm{in}(g',\, t') = (t' + H)^m$. We have $\mathrm{in}(g,\, t) = (t + x_1 H)^m$. Since $0 \neq x_1 H \in Z$, we conclude (h, g, t) is reductive, which contradicts the assumption.
8. We can consider the order $\mathrm{ord}_w : k[[Q]] \to \mathbb{Z}_0 \cup \{\infty\}$ associated with the weight $w = (1, 1, 0, 0, \ldots, 0)$ and the parameter system $Q = \{x_0 = t, x_1, x_2, \ldots, x_n\}$ of $k[[Q]]$. Since $G_j/x_1^j \in Z$ for $1 \leq j \leq m$, we have $\mathrm{ord}_w(g) = \deg(g,\, t) = m$.

Let $\delta = \mathrm{nord}(g)$. By Lemma 2.18.3 we can write

$$g = \left(\prod_{\nu=1}^{\delta}(t - H_\nu)\right) g_s, \quad g_s = t^{m-\delta} + \sum_{j=1}^{m-\delta} L_j t^{m-\delta-j}$$

with some elements $H_1, H_2, \ldots, H_\delta, L_1, L_2, \ldots, L_{m-\delta} \in Z$. We have $\sum_{\nu=1}^{\delta} \mathrm{ord}_w(t - H_\nu) + \mathrm{ord}_w(g_s) = \mathrm{ord}_w(g) = \deg(g,\, t) = \sum_{\nu=1}^{\delta} \deg(t - H_\nu, t) + \deg(g_s,\, t)$. Since $\mathrm{ord}_w(t - H_\nu) \leq \deg(t - H_\nu, t)$ and $\mathrm{ord}_w(g_s) \leq \deg(g_s,\, t)$ by definition, we know $\mathrm{ord}_w(t - H_\nu) = \deg(t - H_\nu, t) = 1$ for $1 \leq \nu \leq \delta$ and $\mathrm{ord}_w(g_s) = \deg(g_s,\, t) = m - \delta$. The last claim is equivalent to the following two conditions hold:

1. $H'_\nu = H_\nu / x_1 \in Z$ for every $1 \leq \nu \leq \delta$.
2. $L'_j = L_j / x_1^j \in Z$ for every $1 \leq j \leq m - \delta$.

Thus we can write

$$g = \left(\prod_{\nu=1}^{\delta}(t + x_1 H'_\nu)\right) g_s, \quad g_s = t^{m-\delta} + \sum_{j=1}^{m-\delta} x_1^j L'_j t^{m-\delta-j}.$$

Substituting $t = (t' + \lambda) x_1$, we obtain

$$g' = \left(\prod_{\nu=1}^{\delta}(t' + \lambda + H'_\nu)\right) g'_s, \quad g'_s = (t' + \lambda)^{m-\delta} + \sum_{j=1}^{m-\delta} L'_j (t' + \lambda)^{m-\delta-j}.$$

Since $\mathrm{ord}(t' + \lambda + H'_\nu) = 0$ or 1 for every $1 \leq \nu \leq \delta$, we have $\mathrm{sord}(g') = \mathrm{sord}(g'_s) \leq \mathrm{ord}(g'_s) \leq m - \delta = \mathrm{ord}(g) - \mathrm{nord}(g) = \mathrm{sord}(g)$. ☐

Theorem 6.2. *Let (B, Q) be a framed space germ, $f \in k[[Q]]$ a non-zero element, and (h, g, t) a Weierstrass representation of f under (B, Q).*

Assume that f does not have a normal crossing, (h, g, t) satisfies the Abhyankar condition, and (h, g, t) is not reductive. Write $Q = \{t, x_1, x_2, \dots, x_n\}$.

1. $\operatorname{ord}(g) \geq 1$ *and* $\operatorname{w}(g,\, t) < \infty$.
2. *Let $b_1, b_2, \dots, b_n$ denote the characteristic numbers of (g, t). There is a number i satisfying $1 \leq i \leq n$ and $b_i > 0$.*
3. *Let i be a number satisfying conditions in 2. Let $C = \{t, x_i\}$. For any choice of an element $x \in C$ and a map $\lambda : C - \{x\} \to k$ there exists a datum (C, x', λ') of a monoidal transformation over (B, Q) such that (C, x', λ') and (C, x, λ) are equivalent and for the elementary monoidal transform (B', Q') of (B, Q) associated with the datum (C, x', λ') there exist elements $h' \in k[[Q']]$, $g' \in k[[Q']]$ $t' \in Q'$ such that one of the following four conditions holds: By $f' \in k[[Q']]$ we denote the image of $f \in k[[Q]]$ by the monoidal transformation $k[[Q]] \to k[[Q']]$.*
   - $\langle\alpha\rangle$ *The element f' has a normal crossing.*
   - $\langle\beta\rangle$ $\operatorname{sord}(f') < \operatorname{sord}(f)$.
   - $\langle\gamma\rangle$ $\operatorname{sord}(f') = \operatorname{sord}(f)$, $\operatorname{nord}(g') < \operatorname{nord}(g)$ *and (h', g', t') is a Weierstrass representation of f' under (B', Q').*
   - $\langle\delta\rangle$ $\operatorname{sord}(f') = \operatorname{sord}(f)$, $\operatorname{nord}(g') = \operatorname{nord}(g)$ *and (h', g', t') is a Weierstrass representation of f' under (B', Q'). Besides, (h', g', t') satisfies the Abhyankar condition, (h', g', t') is not reductive and* $1 \leq \operatorname{w}(g',\, t') = \operatorname{w}(g,\, t) - 1$.

Remark . By assumption and by Lemma 5.2.2 $\operatorname{sord}(f) = \operatorname{sord}(g)$. If (h', g', t') is a semi-Weierstrass representation of f', then $\operatorname{sord}(f') = \operatorname{sord}(g')$ by Lemma 5.2.2. Therefore, in condition $\langle\gamma\rangle$ and $\langle\delta\rangle$, the condition $\operatorname{sord}(f') = \operatorname{sord}(f)$ is equivalent to $\operatorname{sord}(g') = \operatorname{sord}(g)$.

Proof. We write $Z = k[[x_1, x_2, \dots, x_n]]$, and $m = \deg(g,\, t) = \operatorname{ord}(g)$. Let $G_1, G_2, \dots, G_m \in Z$ be the coefficient functions of $(g,\, t)$.
1. If $\operatorname{w}(g,\, t) = \infty$, then f has a normal crossing. We have $m \geq 1$ and $\operatorname{w}(g,\, t) < \infty$.
2. By Lemma 5.5.2.
3. By Definition 5.4.1 $G_j/x_i^j \in Z$ for every $1 \leq j \leq m$.

Exchanging the numbering of x_i's, we can assume that $G_j/x_1^j \in Z$ for every $1 \leq j \leq m$. The assumption in Theorem 6.1 is satisfied. We put $C = \{t, x_1\}$. Let $x \in C$, and let $\lambda : C - \{x\} \to k$ be a map. Let $x' \in C$ denote an element with $C = \{x, x'\}$. We denote $\lambda(x') \in k$ by the same symbol λ.

Case $\mathbf{x = x_1}$. The assumption $\langle B\rangle$ in Theorem 6.1 is satisfied in this case, and we can apply Theorem 6.1. Let (B', Q') be the elementary monoidal transform of (B, Q) associated with the datum $(\{t, x_1\}, x_1, \lambda)$. Let $Q'_1 = Q'$. Let t', g', and h' be the elements in Theorem 6.1.3. By Theorem 6.1.3 (h', g', t') is a semi-Weierstrass representation of f' over $(B', Q') = (B', Q'_1)$. We have $\operatorname{sord}(f') = \operatorname{sord}(g')$.

First, we show that in the case $\operatorname{ord}(g') = \operatorname{ord}(g)$ one of $\langle\alpha\rangle$, $\langle\beta\rangle$, $\langle\delta\rangle$ holds.

By Theorem 6.1.4 (h', g', t') is a Weierstrass representation of f'. By Theorem 6.1.5 and by the assumption we have $\lambda = 0$. By Theorem 6.1.5, 6, 7, (h', g', t') satisfies the Abhyankar condition, (h', g', t') is not reductive, and $\operatorname{w}(g',\, t') = \operatorname{w}(g,\, t) - 1$. By Lemma 2.17.4 we have $\operatorname{w}(g',\, t') \geq 1$.

Assume that $\langle\alpha\rangle$ and $\langle\beta\rangle$ do not hold. Since $\langle\beta\rangle$ does not hold, we have $\mathrm{sord}(f') \geq \mathrm{sord}(f)$, which is equivalent to $\mathrm{sord}(g') \geq \mathrm{sord}(g)$. On the other hand, by Theorem 6.1.8, we have $\mathrm{sord}(g') \leq \mathrm{sord}(g)$. Thus $\mathrm{sord}(g') = \mathrm{sord}(g)$, which is equivalent to $\mathrm{sord}(f') = \mathrm{sord}(f)$. By assumption and Lemma 2.9.2 we obtain $\mathrm{nord}(g') = \mathrm{nord}(g)$.

Second, we show that if $\mathrm{ord}(g') \neq \mathrm{ord}(g)$, then one of $\langle\alpha\rangle$, $\langle\beta\rangle$, $\langle\gamma\rangle$ holds. By Theorem 6.1.4 we have $\mathrm{ord}(g') < \mathrm{ord}(g)$

Assume that $\langle\alpha\rangle$ and $\langle\beta\rangle$ do not hold. By the same argument as above, we obtain $\mathrm{sord}(f') = \mathrm{sord}(g') = \mathrm{sord}(f) = \mathrm{sord}(g)$ by Theorem 6.1.8. Since $\mathrm{ord}(g') < \mathrm{ord}(g)$, we have $\mathrm{nord}(g') = \mathrm{ord}(g') - \mathrm{sord}(g') < \mathrm{ord}(g) - \mathrm{sord}(g) = \mathrm{nord}(g)$.

Below, we will show that there are elements $h'_1, g'_1 \in k[[Q']]$ such that the triplet (h'_1, g'_1, t') is a Weierstrass representation of f' under (B', P') and $\mathrm{nord}(g'_1) \leq \mathrm{nord}(g')$. The triplet (h'_1, g'_1, t') satisfies $\langle\gamma\rangle$.

Let $\delta = \mathrm{nord}(g)$. By the proof of Theorem 6.1.8 we know that there are elements $H''_1, H''_2, \ldots, H''_\delta, L''_1, L''_2, \ldots, L''_{m-\delta} \in Z$ satisfying

$$g' = \left(\prod_{\nu=1}^{\delta}(t' + H''_\nu)\right) g'_s, \quad g'_s = (t')^{m-\delta} + \sum_{j=1}^{m-\delta} L''_j (t')^{m-\delta-j}.$$

We have $\mathrm{sord}(g') \leq \mathrm{sord}(g'_s) \leq \mathrm{ord}(g'_s) \leq m - \delta = \mathrm{sord}(g)$. Since $\mathrm{sord}(g') = \mathrm{sord}(g)$ under our assumption, we have $\mathrm{sord}(g'_s) = \mathrm{ord}(g'_s) = m - \delta = \deg(g'_s,\, t')$. In particular, $(g'_s,\, t')$ is a Weierstrass pair.

Let

$$\begin{aligned} \Delta &= \{\nu \in \mathbb{Z} \mid 1 \leq \nu \leq \delta, H''_\nu \notin M(Z)\} \\ \Gamma &= \{\nu \in \mathbb{Z} \mid 1 \leq \nu \leq \delta, H''_\nu \in M(Z)\} \end{aligned}$$

$$h'_1 = h' \left(\prod_{\nu\in\Delta}(t' + H''_\nu)\right), \quad g'_1 = \left(\prod_{\nu\in\Gamma}(t' + H''_\nu)\right) g'_s.$$

One can check that (h'_1, g'_1, t') is a Weierstrass representation of f' under (B', P'). We have $\mathrm{nord}(g') = \mathrm{nord}(g'_1) + \sum_{\nu\in\Delta} \mathrm{nord}(t' + H''_\nu) = \mathrm{nord}(g'_1)$.

Case $\mathbf{x = t}$. Let (B', Q') be the elementary monoidal transform of (B, Q) associated with the datum $(\{t, x_1\}, t, \lambda)$. We identify elements in B and their images in B' by the monoidal transformation.

We consider the case $\lambda = 0$ first. Let $t' = t$, $x'_1 = x_1/t$ and $x'_i = x_i$ for $2 \leq i \leq n$. We have $Q' = \{t', x'_1, x'_2, \ldots, x'_n\}$, and $k[[Q']] = k[[t', x'_1, x'_2, \ldots, x'_n]] \supset k[[Q]] = k[[t, x_1, x_2, \ldots, x_n]] \supset Z$.

$$g = t^m + \sum_{j=1}^{m} \frac{G_j}{x_1^j} \cdot x_1^j t^{m-j} = (t')^m \left\{ 1 + \sum_{j=1}^{m} \frac{G_j}{x_1^j}(x'_1)^j \right\}$$

Thus $g/(t')^m \in k[[Q']]^*$. On the other hand,

$$\frac{h}{x_1^{s_1} x_2^{s_2} \cdots x_n^{s_n}} = \frac{h}{(t')^{s_1}(x'_1)^{s_1}(x'_2)^{s_2}\cdots(x'_n)^{s_n}} \in k[[Q]]^* \subset k[[Q']]^*,$$

where $s_1, s_2, \ldots, s_n$ are non-negative integers in Definition 5.1.2. Thus $f' = f = hg$ has a normal crossing in $k[[Q']]$. The condition $\langle\alpha\rangle$ holds in this case.

Assume that $\lambda \neq 0$. Let (B'_1, Q'_1) be the elementary monoidal transform of (B, Q) associated with the datum $(\{t, x_1\}, x_1, 1/\lambda)$. Data $(\{t, x_1\}, t, \lambda)$ and $(\{t, x_1\}, x_1, 1/\lambda)$

are equivalent. We have treated (B_1', Q_1') in the above case $x = x_1$. We obtain the claim by the above case. $\square$

Let (B, P) be a framed space germ over k, and (h, g, t) a Weierstrass representation over (B, P). We write $P = \{x_1, x_2, \dots, x_n\}$.

We consider the *sixth framed resolution game* characterized by the conditions below:

- The number of the data: $\sharp P + 3$.
- The initial conditions: Let $(h, g, t, \xi_1, \xi_2, \dots, \xi_n)$ denote the data.
  1. (h, g, t) is a Weierstrass representation over (B, P).
  2. $P = \{\xi_1, \xi_2, \dots, \xi_n\}$.
- The winning rules:
  1. There exist an element $H \in k[[P]]$ such that the triplet $(h, g, t + H)$ is a Weierstrass representation over (B, P) satisfying the Abhyankar condition and $(h, g, t + H)$ is not reductive.
  2. The product $\xi_1 \xi_2 \cdots \xi_n \in k[[P]]$ has a normal crossing with respect to P.

We consider the following statement $AN(B, P)$ for (B, P).

$\mathbf{AN(B, P)}$: For every separable Weierstrass representation (h, g, t) over (B, P) the player $\langle I \rangle$ can always win the game just above.

Recall the statement $RW(B, Q)$ we gave before Lemma 3.14.

Theorem 6.3. *Let r be a positive integer. Assume that $AN(B, P)$ is true for every framed space germ (B, P) with $\sharp P \leq r$.*

Then, $RW(B, Q)$ is true for every framed space germ (B, Q) with $\sharp Q \leq r + 1$.

Proof. Let (B, Q) be a framed space germ over k with $\sharp Q \leq r + 1$. Let $f \in k[[Q]]$ be an arbitrary non-zero element. Consider the first framed resolution game after Lemma 3.13 for $\ell = 1$ and $f_1 = f$. We follow the procedures described below in numerical order, and we go back to the line with a number when we encounter a sentence telling it.

1. If f has a normal crossing in B, then by replacing Q by an appropriate Q_1 $\langle I \rangle$ can win the first game.

   Below we can assume moreover that f does not have a normal crossing in B.
2. At a step of the game, by Lemma 5.2.1 the player $\langle I \rangle$ can choose a parameter system Q_1 of $k[[Q]]$ such that there is a separable Weierstrass representation (h, g, t) of f under (B, Q_1). Thus, after one step, we can assume that we have a separable Weierstrass representation (h, g, t) of f under (B, Q).
3. Let $P = Q - \{t\}$. Obviously P is a frame with $\sharp P \leq r$ and (h, g, t) is a separable Weierstrass representation of f over (B, P). Note that each step of the sixth game corresponding to (h, g, t) can be regarded as a step of the first game corresponding to $f = hg$, since $h'g'$ is equal to the image of f by $B \to B'$. If $\langle I \rangle$ replaces P by P_1 in the sixth game, then we replace $Q = P \cup \{t\}$ by $Q_1 = P_1 \cup \{t\}$ in the first game. Thus, by assumption, after a finite number of steps we can assume that $Q = P \cup \{t\}$ and that there exist an element $H \in k[[P]]$ such that the triplet $(h, g, t + H)$ is a Weierstrass representation of f over (B, P) such that it satisfies the Abhyankar condition and it is not reductive.

By replacing $Q = P \cup \{t\}$ by $Q_1 = P \cup \{t + H\}$, we can assume that (h, g, t) is a Weierstrass representation of f under (P, Q) such that it satisfies the Abhyankar condition and it is not reductive.

We consider the case $\mathrm{w}(g,\, t) = \infty$. In this case we can write $f = ht^m$ where $m = \deg(g,\, t)$. Thus f has a normal crossing with respect to Q, which contradicts our assumption.

We can assume that f does not have a normal crossing in B and that we have a Weierstrass representation (h, g, t) of f under (B, Q) such that it satisfies the Abhyankar condition, and it is not reductive.

4. At the next step of the game the player $\langle I\rangle$ chooses a subset $C \subset Q$ with $\sharp C = 2$ described in Theorem 6.2. After $\langle II\rangle$ chooses $x \in C$ and $\lambda : C - \{x\} \to k$ we conclude a step of the first game by replacement. Moreover, after $\langle I\rangle$ chooses an appropriate Q_1 and we conclude a step by replacement. Here by $(\bar{B}, \bar{Q})$ we denote the framed space germ before placements twice. By $\bar{f} \in k[[\bar{Q}]]$, $\bar{g} \in k[[\bar{Q}]]$, $\bar{h} \in k[[\bar{Q}]]$ and $\bar{t} \in \bar{Q}$ we denote the function under consideration before placements twice. The image of $\bar{f}$ by $k[[\bar{Q}]] \to k[[Q]]$ coincides with f. By Theorem 6.2 we have elements $g \in k[[Q]]$, $h \in k[[Q]]$ and $t \in Q$ (Note that g, h and t do not coincide with the images of $\bar{g}$, $\bar{h}$ and $\bar{t}$.), and we can assume that one of the four cases $\langle\alpha\rangle$, $\langle\beta\rangle$, $\langle\gamma\rangle$, $\langle\delta\rangle$ in Theorem 6.2 holds.

   Assume that case $\langle\alpha\rangle$ holds. Then, f has a normal crossing. We go back to 1 above. The player $\langle I\rangle$ can win the game.

   Assume that $\langle\beta\rangle$ holds. $\mathrm{sord}(f) < \mathrm{sord}(\bar{f})$. In this case we go back to 2 above.

   Assume that $\langle\gamma\rangle$ holds. $\mathrm{sord}(f) = \mathrm{sord}(\bar{f})$, $\mathrm{nord}(g) < \mathrm{nord}(\bar{g})$ and (h, g, t) is a Weierstrass representation of f under (B, Q). We go back to 3 above with a Weierstrass representation (h, g, t).

   Assume that $\langle\delta\rangle$ holds. $\mathrm{sord}(f) = \mathrm{sord}(\bar{f})$, $\mathrm{nord}(g) = \mathrm{nord}(\bar{g})$ and (h, g, t) is a Weierstrass representation of f under (B, Q). Besides, (h, g, t) satisfies the Abhyankar condition, (h, g, t) is not reductive and $1 \leq \mathrm{w}(g,\, t) = \mathrm{w}(\bar{g},\, \bar{t}) - 1$. We go back to 4 above with a Weierstrass representation (h, g, t).

We have to show that the described procedure terminates in finite steps.

We consider the totally ordered abelian group $\mathbb{Z}^3$ with the lexicographic order. Note that when we enter 4, we have a Weierstrass representation (h, g, t) of f satisfying strong conditions, and we can associate the element

$$c(h, g, t) = (\mathrm{sord}(g), \mathrm{nord}(g), [\mathrm{w}(g,\, t)]) \in \mathbb{Z}^3,$$

where [ ] denotes the Gauss symbol. Note that $\mathrm{sord}(g) \geq 0$, $\mathrm{nord}(g) \geq 0$ and $[\mathrm{w}(g,\, t)] \geq 1$ by definition. Thus $c(h, g, t) \in \mathbb{Z}_0^3$

In the middle of 4 we obtain framed space germs (B, Q), $(\bar{B}, \bar{Q})$ and elements f, h, g, t, $\bar{f}$, $\bar{h}$, $\bar{g}$ and $\bar{t}$. Then, we go back to 1, 2, 3 or 4 and encounter a lot of replacements. Though we repeat replacements, we fix notations in the middle of 4 for explanation.

Assume that we have $\langle\beta\rangle$. Going back to 2, we have a parameter system Q_1 of $k[[Q]]$ and a Weierstrass representation (h_1, g_1, t_1) of f under (B, Q_1). We have $\mathrm{sord}(g_1) = \mathrm{sord}(f)$. After entering 3 at the end of 3, we have an IAMT (B', Q') of (B, Q_1) and a Weierstrass representation (h', g', t') of f' under (B', Q'), where f' denotes the image of f by $k[[Q_1]] \to k[[Q']]$. Since g' is the image of g_1 by $k[[Q_1]] \to k[[Q']]$, we have $\mathrm{sord}(g') \leq \mathrm{sord}(g_1)$ by Lemma 5.3.5. We have $\mathrm{sord}(g') <$

$\mathrm{sord}(\bar{f}) = \mathrm{sord}(\bar{g})$, and $c(h', g', t') < c(\bar{h}, \bar{g}, \bar{t})$. We know that when we enter 4 again, we have a smaller value of $c(h, g, t)$.

Assume that we have $\langle\gamma\rangle$. Going back to 3, at the end of 3 we have an IAMT (B', Q') of (B, Q) and a Weierstrass representation $(h', g', t' + H)$ of f' under (B', Q'), where f', h', g', and t' denote the images of f, h, g, and t by $k[[Q]] \to k[[Q']]$ respectively. We have $\mathrm{sord}(g') \leq \mathrm{sord}(g) = \mathrm{sord}(f) = \mathrm{sord}(\bar{f}) = \mathrm{sord}(\bar{g})$, $\mathrm{nord}(g) < \mathrm{nord}(\bar{g})$, and $\mathrm{sord}(g') + \mathrm{nord}(g') = \mathrm{ord}(g') = \deg(g', t' + H) = \deg(g', t') = \deg(g, t) = \mathrm{ord}(g) = \mathrm{sord}(g) + \mathrm{nord}(g)$. If $\mathrm{sord}(g') < \mathrm{sord}(g)$, we have $c(h', g', t') < c(\bar{h}, \bar{g}, \bar{t})$. Otherwise, we have $\mathrm{sord}(g') = \mathrm{sord}(\bar{g})$, $\mathrm{nord}(g') = \mathrm{nord}(g) < \mathrm{nord}(\bar{g})$, and again we have $c(h', g', t') < c(\bar{h}, \bar{g}, \bar{t})$.

Assume that we have $\langle\delta\rangle$. Since $\mathrm{sord}(g) = \mathrm{sord}(f) = \mathrm{sord}(\bar{f}) = \mathrm{sord}(\bar{g})$, we have $c(h, g, t) < c(\bar{h}, \bar{g}, \bar{t})$.

In any case we know that when we enter 4 again, we have a smaller value of $c(h, g, t)$. Since there is no infinite sequence $\{c_i \in \mathbb{Z}_0^3 \mid i \in \mathbb{Z}_0\}$ with $c_i > c_{i+1}$ for any $i \in \mathbb{Z}_0$, the above procedure has to terminate in finite steps. $\square$

By Theorem 6.3 it turns out that the problem below is essential.

Problem: Let r be a positive integer. Show that $AN(B, P)$ is true for every framed space germ (B, P) with $\sharp\, P \leq r$, assuming that $RW(B, P)$ is true for every framed space germ with $\sharp\, P \leq r$.

If we could solve Problem, then $RW(B, Q)$ would be true for *every* (B, Q). Note that if we drop the last condition "$(h', g', t' + H')$ is not reductive" from $AN(B, P)$, then Corollary 5.9 solves Problem.

In the remaining part of this paper, we consider Problem. The concept of reduction sequences in Section 7 would be effective.

Remark . In the essential parts until here, we have never used the assumption that the characteristic number p of the ground field k is positive.

If the characteristic number of the ground field k is zero, then we can easily solve Problem.

Let (g, t) be a Weierstrass pair with $\mathrm{w}(g,\, t) < \infty$, and $G_1, G_2, \ldots, G_m$ the coefficient functions of (g, t). If the characteristic number is zero, then we can make the parallel coordinate transform of (g, t) along $H = G_1/m$. As the result, we can assume that (g, t) is a Weierstrass pair with $G_1 = 0$ from the beginning. It is easy to see that if $G_1 = 0$ and if the characteristic number is zero, then any pull-back (g', t') of (g, t) is not reductive. Thus, by Corollary 5.9 Problem can be solved.

Any hypersurface singularity of a germ of a variety in characteristic zero can be resolved by iterated monoidal transformations.

In Abhyankar [2] the parallel coordinate transformation along $H = G_1/m$ is called the Shreedharacharya transformation.

7. Reduction sequence

We introduce an effective tool to treat the Abhyankar condition and the condition "not reductive" simultaneously for a Weierstrass representation.

Definition 7.1. Let (B, P) be a framed space germ over k, (h, g, t) a Weierstrass representation over (B, P), and ω either a positive integer or symbol ∞. We consider a sequence

$$\mathcal{S} = \{(B_\nu, P_\nu, H_\nu) \mid 1 \leq \nu < \omega + 1\}$$

of triplets (B_ν, P_ν, H_ν) numbered by integers ν with $1 \leq \nu < \omega + 1$. Let $(B_0, P_0) = (B, P)$ and $(h_0, g_0, t_0) = (h, g, t)$. If $\mathcal{S}$ satisfies the following conditions 0–2 for every integer ν with $1 \leq \nu < \omega + 1$, the condition 3 for every integer ν with $1 \leq \nu < \omega$, and the condition 4, then we call $\mathcal{S}$ a *reduction sequence* of (h, g, t). We call ω the *length* of $\mathcal{S}$.

0. (B_ν, P_ν) is a framed space germ over k. $H_\nu \in k[[P_\nu]]$.
1. (B_ν, P_ν) is an IAMT of $(B_{\nu-1}, P_{\nu-1})$ with a normal crossing.

To state conditions 2–4 we define two Weierstrass representations $(h'_{\nu-1}, g'_{\nu-1}, t'_{\nu-1})$ and (h_ν, g_ν, t_ν) over (B_ν, P_ν) by the following conditions $\langle A\rangle$ and $\langle B\rangle$:

$\langle A\rangle$ $(h'_{\nu-1}, g'_{\nu-1}, t'_{\nu-1})$ is the pull-back of the Weierstrass representation $(h_{\nu-1}, g_{\nu-1}, t_{\nu-1})$ over $(B_{\nu-1}, P_{\nu-1})$ by the IMT $(B_{\nu-1}, P_{\nu-1}) \to (B_\nu, P_\nu)$.

$\langle B\rangle$ $(h_\nu, g_\nu, t_\nu) = (h'_{\nu-1}, g'_{\nu-1}, t'_{\nu-1} + H_\nu)$.

2. $(h'_{\nu-1}, g'_{\nu-1}, t'_{\nu-1})$ satisfies the Abhyankar condition.
3. The element $H_\nu \in k[[P_\nu]]$ is an effective reducing element of $(g'_{\nu-1}, t'_{\nu-1})$.
4. If $(h'_{\omega-1}, g'_{\omega-1}, t'_{\omega-1})$ is not reductive, then $H_\omega = 0$. If $(h'_{\omega-1}, g'_{\omega-1}, t'_{\omega-1})$ is reductive, then $\mathrm{w}(g_\omega, t_\omega) = \infty$.

Besides, if ω is a positive integer and if $\mathcal{S}$ satisfies the conditions 0–3 for every integer ν with $1 \leq \nu < \omega + 1$, then we say that $\mathcal{S}$ is a *semi-reduction sequence* of (h, g, t) of *length* ω. We call the empty set $\mathcal{S} = \emptyset = \{\}$ *semi-reduction sequence of length* 0.

We call the above sequence $\{(h_\nu, g_\nu, t_\nu) \mid 0 \leq \nu < \omega + 1\}$ of Weierstrass representations the *transform Weierstrass representation sequence* of (h, g, t) along $\mathcal{S}$.

If $\omega = \infty$, then we say that $\mathcal{S}$ has *infinite length.* If $\omega < \infty$, then we say that $\mathcal{S}$ has *finite length.*

Remark . 1. The length of a reduction sequence is either a positive integer or ∞. The length of a semi-reduction sequence is either a positive integer or 0.
2. For simplicity, sometimes we identify an element in B_μ and the image of it in B_ν for any pair μ, ν of integers with $0 \leq \mu \leq \nu < \omega + 1$.
3. Under the identification in 2, $h = h_\nu$, $g = g_\nu$ and $t_\nu = t + H_1 + H_2 + \cdots + H_\nu$ for any $0 \leq \nu < \omega + 1$.
4. Let $\mathcal{S}$ be a reduction sequence of (h, g, t) with finite length ω. The last Weierstrass representation $(h_\omega, g_\omega, t_\omega)$ satisfies the Abhyankar condition, and is not reductive.
5. By condition 3 of Definition 7.1, for every $1 \leq \nu < \omega$ $(h'_{\nu-1}, g'_{\nu-1}, t'_{\nu-1})$ is reductive. If $\mathcal{S}$ is a semi-reduction sequence, $(h'_{\omega-1}, g'_{\omega-1}, t'_{\omega-1})$ is also reductive.
6. According to the above condition in 3, in the case that ω is a positive integer, H_ω is not necessarily an effective reducing element of $(g'_{\omega-1}, t'_{\omega-1})$.

   If $H_\omega = 0$, then H_ω is never an effective reducing element of $(g'_{\omega-1}, t'_{\omega-1})$.
7. If $\mathcal{S} = \{(B_\nu, P_\nu, H_\nu) \mid 1 \leq \nu < \omega + 1\}$ is a reduction sequence, then for every integer λ with $1 \leq \lambda < \omega$, the subsequence $\mathcal{S}_\lambda = \{(B_\nu, P_\nu, H_\nu) \mid 1 \leq \nu \leq \lambda\}$ of $\mathcal{S}$ is a semi-reduction sequence.

   For a given infinite sequence of quadruples $\mathcal{S} = \{(B_\nu, P_\nu, H_\nu) \mid 1 \leq \nu < \infty\}$, $\mathcal{S}$ is a reduction sequence of (h, g, t) with infinite length, if and only if, for any positive integer λ the subsequence $\mathcal{S}_\lambda = \{(B_\nu, P_\nu, H_\nu) \mid 1 \leq \nu \leq \lambda\}$ of $\mathcal{S}$ is a semi-reduction sequence of (h, g, t).

Recall the statement $RW(B, P)$ we gave before Lemma 3.14.

Lemma 7.2. *Let r be a positive integer. Assume that $RW(B,P)$ is true for every framed space germ (B,P) such that $\sharp\, P \leq r$.*

Let (B,P) be a framed space germ over k with $\sharp\, P \leq r$, and (h,g,t) a Weierstrass representation over (B,P).

We consider the seventh framed resolution game *characterized by the conditions below:*

- *The number of the data:* 3.
- *The initial conditions: The data (h,g,t) is a Weierstrass representation over (B,P).*
- *The winning rules: The product hg has a normal crossing in B.*

The player $\langle I\rangle$ of this resolution game can choose a strongly increasing sequence $\{i_\nu \mid 1 \leq \nu < \omega+1\}$ of positive integers with the following property: Denote the framed space germ obtained after the i-th step of the game by (B_i,P_i). There exists an element $H_\nu \in k[[P_{i_\nu}]]$ for every ν such that $\mathcal{S} = \{(B_{i_\nu}, P_{i_\nu}, H_\nu) \mid 1 \leq \nu < \omega+1\}$ is a reduction sequence of (h,g,t).

Proof. The empty set $\mathcal{S}_0 = \{\}$ is a semi-reduction sequence of (h,g,t) with length 0. Let $i_0 = 0$. Thus by induction, we can assume that we have obtained a strongly increasing sequence $\{i_\nu \mid 0 \leq \nu \leq \lambda\}$ of positive integers for some non-negative integer λ, the i_λ-th step of the game has terminated and we already have a semi-reduction sequence $\mathcal{S}_\lambda = \{(B_{i_\nu}, P_{i_\nu}, H_\nu) \mid 1 \leq \nu \leq \lambda\}$ of (h,g,t) with length λ. Let $(B_0,P_0) = (B,P)$, $(h_0,g_0,t_0) = (h,g,t)$, and $\{(h_\nu, g_\nu, t_\nu) \mid 0 \leq \nu \leq \lambda\}$ be the transform Weierstrass representation sequence along $\mathcal{S}_\lambda$. Since $(B_{i_\lambda}, P_{i_\lambda})$ is an IAMT of $(B_0,P_0) = (B,P)$, we have $\sharp\, P_{i_\lambda} = \sharp\, P \leq r$. Thus $RW(B_{i_\lambda}, P_{i_\lambda})$ is true by assumption.

By Corollary 5.9 there exists an integer $i_{\lambda+1}$ with $i_{\lambda+1} > i_\lambda$ such that there exists an IAMT $(B_{i_{\lambda+1}}, P_{i_{\lambda+1}})$ of $(B_{i_\lambda}, P_{i_\lambda})$ with a normal crossing such that the pull-back $(h'_\lambda, g'_\lambda, t'_\lambda)$ of $(h_\lambda, g_\lambda, t_\lambda)$ by the IMT $(B_{i_\lambda}, P_{i_\lambda}) \to (B_{i_{\lambda+1}}, P_{i_{\lambda+1}})$ satisfies the Abhyankar condition, if they continue the game.

In case where $(h'_\lambda, g'_\lambda, t'_\lambda)$ is not reductive, letting $H_{\lambda+1} = 0$, $\mathcal{S} = \{(B_{i_\nu}, P_{i_\nu}, H_\nu) \mid 1 \leq \nu \leq \lambda+1\}$, we have a reduction sequence $\mathcal{S}$ of (h,g,t) with length $\lambda+1$.

In case where $(h'_\lambda, g'_\lambda, t'_\lambda)$ is reductive, by Lemma 5.15.7 there exists an effective reducing element $H_{\lambda+1} \in k[[P_{i_{\lambda+1}}]]$, and it satisfies $\mathrm{w}(g'_\lambda,\, t'_\lambda) < \mathrm{w}(g'_\lambda,\, t'_\lambda + H_{\lambda+1})$.

If $\mathrm{w}(g'_\lambda,\, t'_\lambda + H_{\lambda+1}) = \infty$, letting $\mathcal{S} = \{(B_{i_\nu}, P_{i_\nu}, H_\nu) \mid 1 \leq \nu \leq \lambda+1\}$, we have a reduction sequence $\mathcal{S}$ of (h,g,t) with length $\lambda+1$.

In case where $\mathrm{w}(g'_\lambda,\, t'_\lambda + H_{\lambda+1}) < \infty$, $\mathcal{S}_{\lambda+1} = \{(B_{i_\nu}, P_{i_\nu}, H_\nu) \mid 1 \leq \nu \leq \lambda+1\}$ is a semi-reduction sequence of (h,g,t) with length $\lambda+1$, and $\mathcal{S}_{\lambda+1}$ is an extension of $\mathcal{S}_\lambda$. We can ascend an induction step.

Assume that for any non-negative integer λ a reduction sequence $\mathcal{S}$ of (h,g,t) with length $\lambda+1$ is not constructed, when we repeat the above procedure. Then, for any non-negative integer λ a semi-reduction sequence $\mathcal{S}_\lambda$ of (h,g,t) with length λ is constructed. For any pair μ, λ of integers with $0 \leq \mu < \lambda$ $\mathcal{S}_\lambda$ is an extension of $\mathcal{S}_\mu$. In this case $\mathcal{S} = \{(B_\nu, P_\nu, H_\nu) \mid 1 \leq \nu < \infty\}$ is a reduction sequence of (h,g,t) with infinite length. □

Below we consider reduction sequences with infinite length.

Proposition 7.3. *Let (B,P) be a framed space germ over k, (h,g,t) a Weierstrass representation over (B,P), and $\mathcal{S} = \{(B_\nu, P_\nu, H_\nu) \mid 1 \leq \nu < \infty\}$ an reduction sequence of (h,g,t) with infinite length. Let $(B_0,P_0) = (B,P)$ and $(h_0,g_0,t_0) =$*

(h, g, t). Let $\{(h_\nu, g_\nu, t_\nu) \mid 0 \le \nu < \omega + 1\}$ be the transform Weierstrass representation sequence of (h, g, t). We denote $m = \deg(g,\, t)$, $P_\nu = \{x_{\nu,1}, x_{\nu,2}, \dots, x_{\nu,n}\}$, and $Z_\nu = k[[P_\nu]] = k[[x_{\nu,1}, x_{\nu,2}, \dots, x_{\nu,n}]]$. By the IMT $B_\nu \to B_{\nu+1}$ we define an inductive system $\{B_\nu \mid \nu \ge 0\}$, and denote $\mathcal{B} = \varinjlim{}_\nu B_\nu = \cup_{\nu=0}^{\infty} B_\nu$. For simplicity, we identify B_ν and the image of the canonical homomorphism $B_\nu \to \mathcal{B}$. Under this identification, we have $h = h_\nu$, $g = g_\nu$ and $t_\nu = t + \sum_{\mu=1}^{\nu} H_\mu$ for every non-negative integer ν. We understand $\sum_{\mu=1}^{0} H_\mu = 0$ here as a convention. Also we denote $\mathcal{Z} = \varinjlim{}_\nu Z_\nu = \cup_{\nu=0}^{\infty} Z_\nu$. Note that $\mathrm{w}(g,\, t) < \infty$ and $m \ge 1$, since $\mathcal{S}$ with infinite length exists.

Then, there exist an infinite sequence

$$\{(b_{\nu,1}, b_{\nu,2}, \dots, b_{\nu,n}) \mid 1 \le \nu < \infty\}$$

of n-tuples $(b_{\nu,1}, b_{\nu,2}, \dots, b_{\nu,n})$ of non-negative integers $b_{\nu,i}$, an infinite sequence

$$\{\eta_\nu \mid 1 \le \nu < \infty\}$$

of elements $\eta_\nu \in \mathcal{Z}$, and an infinite sequence

$$\{(F_{\nu,1}, F_{\nu,2}, \dots, F_{\nu,m}) \mid 0 \le \nu < \infty\}$$

of m-tuples $(F_{\nu,1}, F_{\nu,2}, \dots, F_{\nu,m})$ of elements $F_{\nu,j} \in \mathcal{Z}$ satisfying the following conditions $(1.\nu)$, $(2.\nu)$, $(3.\nu)$ for every positive integer ν and the condition $(4.\nu)$ for every non negative integer ν. For simplicity, we write

$$x_\nu^{b_\nu} = x_{\nu,1}^{b_{\nu,1}} x_{\nu,2}^{b_{\nu,2}} \cdots x_{\nu,n}^{b_{\nu,n}} = \prod_{i=1}^{n} x_{\nu,i}^{b_{\nu,i}}.$$

$(1.\nu)$ *Note that $F_{\nu-1,j} \in Z_{\nu-1} \subset Z_\nu$ for every $1 \le j \le m$. The m-tuple $(F_{\nu-1,1}, F_{\nu-1,2}, \dots, F_{\nu-1,m})$ of elements in Z_ν satisfy the Abhyankar condition over the framed space germ (B_ν, P_ν). The characteristic numbers of $(F_{\nu-1,1}, F_{\nu-1,2}, \dots, F_{\nu-1,m})$ are equal to $(b_{\nu,1}, b_{\nu,2}, \dots, b_{\nu,n})$. $\mathrm{w}(F_{\nu-1,1}, F_{\nu-1,2}, \dots, F_{\nu-1,m}) = \sum_{i=1}^{n} b_{\nu,i}$.*

$(2.\nu)$ *At least one of $b_{\nu,1}, b_{\nu,2}, \dots, b_{\nu,n}$ is not zero. $x_\nu^{b_\nu} \in M(Z_\nu)$.*

$(3.\nu)$

$$H_\nu = \eta_\nu \prod_{\mu=1}^{\nu} x_\mu^{b_\mu}, \quad \eta_\nu \in Z_\nu^*.$$

$(4.\nu)$ *$F_{\nu,j} \in M(Z_\nu)$ for every $1 \le j \le m$.*

$$g = t_\nu^m + \sum_{j=1}^{m} \left(\prod_{\mu=1}^{\nu} x_\mu^{b_\mu} \right)^j F_{\nu,j} t_\nu^{m-j}.$$

We understand $\prod_{\mu=1}^{0} x_\mu^{b_\mu} = 1$ here as a convention.

Proof. By induction on ν we define $F_{\nu,j}$ $(1 \le j \le m)$, $b_{\nu,i}$ $(1 \le i \le n)$, and η_ν.

First, let $F_{0,1}, F_{0,2}, \dots, F_{0,m} \in M(Z_0) = M(Z)$ be the coefficient functions of $(g_0, t_0) = (g, t)$. The condition (4.0) holds.

Let ν be a positive integer. Assume that in addition to $F_{0,j}$ $(1 \le j \le m)$ which we have just defined, we have already defined $F_{\mu,j}$, $b_{\mu,i}$, η_μ for every integers μ, j, i satisfying $1 \le \mu < \nu$, $1 \le j \le m$, $1 \le i \le n$, and they satisfy (4.0) and $(1.\mu)$, $(2.\mu)$, $(3.\mu)$ $(4.\mu)$ for every positive integer μ with $1 \le \mu < \nu$.

We will define $b_{\nu,1}, b_{\nu,2}, \dots, b_{\nu,n}$, η_ν, and $(F_{\nu,1}, F_{\nu,2}, \dots, F_{\nu,m})$ so that they will satisfy $(1.\nu)$, $(2.\nu)$ $(3.\nu)$ and $(4.\nu)$.

By definition of the reduction sequence, the Weierstrass representation $(h, g, t_{\nu-1})$ over (B_ν, P_ν) satisfies the Abhyankar condition and is reductive. By $(4.\nu-1)$ the coefficient functions of $(g, t_{\nu-1})$ is equal to $(\prod_{\mu=1}^{\nu-1} x_\mu^{b_\mu})^j F_{\nu-1,j}$ $1 \le j \le m$. Since the IMT $(B_\mu, P_\mu) \to (B_\nu, P_\nu)$ for any $0 \le \mu \le \nu$ has a normal crossing, we have non-negative integers $c_1, c_2, \ldots, c_n$ and an element $v \in Z_\nu^*$ with

$$\prod_{\mu=1}^{\nu-1} x_\mu^{b_\mu} = v x_{\nu,1}^{c_1} x_{\nu,2}^{c_2} \cdots x_{\nu,n}^{c_n}.$$

By Lemma 5.15.9 the m-tuple $v^j F_{\nu-1,j}$ $1 \le j \le m$ satisfies the Abhyankar condition, and the weight of it is equal to the sum of the characteristic numbers of it.

By Lemma 5.14.3 also the m-tuple $F_{\nu-1,j}$ $1 \le j \le m$ satisfies the Abhyankar condition. Besides, the m-tuple $v^j F_{\nu-1,j}$ $1 \le j \le m$ and the m-tuple $F_{\nu-1,j}$ $1 \le j \le m$ have common weight and common characteristic numbers.

Let $b_{\nu,1}, b_{\nu,2}, \ldots, b_{\nu,n}$ be the characteristic numbers of the m-tuple $F_{\nu-1,j}$ $1 \le j \le m$. By Lemma 5.15.9 one knows that they satisfy $(1.\nu)$ and $(2.\nu)$.

By Lemma 5.15.9 the characteristic numbers of $(g, t_{\nu-1})$ are equal to $c_i + b_{\nu,i}$ $1 \le i \le n$. By definition of $x_\nu^{b_\nu}$ we have

$$x_{\nu,1}^{c_1+b_{\nu,1}} x_{\nu,2}^{c_2+b_{\nu,2}} \cdots x_{\nu,n}^{c_n+b_{\nu,n}} = \frac{1}{v} \prod_{\mu=1}^{\nu} x_\mu^{b_\mu}.$$

Since H_ν is an effective reducing element of $(g, t_{\nu-1})$, we have

$$\eta = \frac{H_\nu}{x_{\nu,1}^{c_1+b_{\nu,1}} x_{\nu,2}^{c_2+b_{\nu,2}} \cdots x_{\nu,n}^{c_n+b_{\nu,n}}} = \frac{v H_\nu}{\prod_{\mu=1}^{\nu} x_\mu^{b_\mu}} \in Z_\nu^*,$$

and $\mathrm{w}(g,\, t_{\nu-1}) < \mathrm{w}(g,\, t_{\nu-1} + H_\nu)$. Thus $\eta_\nu = \eta / v$ satisfies $(3.\nu)$.

Since $t_\nu = t_{\nu-1} + H_\nu$, by Lemma 5.15.8 there exists an m-tuple $F_j \in M(Z_\nu)$ $1 \le j \le m$ satisfying

$$g = t_\nu^m + \sum_{j=1}^{m} (x_{\nu,1}^{c_1+b_{\nu,1}} x_{\nu,2}^{c_2+b_{\nu,2}} \cdots x_{\nu,n}^{c_n+b_{\nu,n}})^j F_j t_\nu^{m-j}.$$

Let $F_{\nu,j} = F_j / v^j$ for $1 \le j \le m$. The m-tuple $F_{\nu,j}$ $1 \le j \le m$ satisfies $(4.\nu)$. ☐

Corollary 7.4. 1. *For every positive integer* ν *also the following* $(5.\nu)$ *holds:* $(5.\nu)$ *For every* $1 \le j \le m$

$$u_{\nu,j} = \frac{F_{\nu-1,j}}{(x_\nu^{b_\nu})^j} \in Z_\nu, \;\; u_{\nu,j}(0) = \binom{m}{j} \eta_\nu(0)^j, \;\; \textit{and}$$

$$g = t_{\nu-1}^m + \sum_{j=1}^{m} \left(\prod_{\mu=1}^{\nu} x_\mu^{b_\mu} \right)^j u_{\nu,j} t_{\nu-1}^{m-j}.$$

2. *For every positive integer* ν *and for every* $1 \le j \le m$

$$(x_\nu^{b_\nu})^j F_{\nu,j} = \binom{m}{m-j} (-\eta_\nu x_\nu^{b_\nu})^j + \sum_{i=1}^{j} \binom{m-i}{m-j} (-\eta_\nu x_\nu^{b_\nu})^{j-i} F_{\nu-1,i}.$$

Proof. 1. By (1.ν) we have $u_{\nu,j} \in Z_\nu$. Substituting $F_{\nu-1,j} = (x_\nu^{b_\nu})^j u_{\nu,j}$ into (4.$\nu - 1$), we obtain the equality for g.

For simplicity, we write $X = x_{\nu,1}^{c_1+b_{\nu,1}} x_{\nu,2}^{c_2+b_{\nu,2}} \cdots x_{\nu,n}^{c_n+b_{\nu,n}}$. By the equality we just obtained, one knows

$$\mathrm{in}(g, t_{\nu-1}) = t_{\nu-1}^m + \sum_{j=1}^{m} (v(0)X)^j u_{\nu,j}(0) t_{\nu-1}^{m-j}.$$

By (3.ν) $\mathrm{in}(H_\nu) = \eta_\nu(0)v(0)X$. By Lemma 5.12.3 $\mathrm{in}(g, t_{\nu-1}) = (t_{\nu-1} + \mathrm{in}(H_\nu))^m$. Thus for every $1 \le j \le m$

$$(v(0)X)^j u_{\nu,j}(0) = \binom{m}{j} (v(0)X)^j \eta_\nu(0)^j.$$

Since $v(0)X \neq 0$, we obtain the desired equality.
2. It follows from Lemma 5.10.2. □

8. Proof of Theorem 1.1

We give a proof of Theorem 1.1 in Introduction.

Proof. Assume that $AN(B,P)$ is true for every framed space germ (B,P) with $\dim B = n+1$ and $\sharp P \le n$.

Let B be a space germ with $\dim B = n+1$, $f \in B$ a non-zero element, and Q a parameter system of B. One has $f \in k[[Q]] = B$ and $\sharp Q = n+1$. By Theorem 6.3 $RW(B,Q)$ is true. This implies that $(B,f) \notin \mathcal{X}$ and $\min\{\dim B \mid (B,f) \in \mathcal{X}\} > n+1$, which contradicts the definition of the integer n.

We conclude that there exists a framed space germ (B,P) such that $\dim B = n+1$, $\sharp P \le n$ and $AN(B,P)$ is false. Below by (B,P) we denote the framed space germ with these properties.

Note that $RW(B,P)$ is true. Let $A = k[[P]]$ and $f \in A$ an arbitrary non-zero element. Since $\dim A = \sharp P \le n$, the element $f \in A$ can be resolved. We consider the resolution game of $f \in A$. Assume that the player $\langle I \rangle$ chooses a non-singular closed subscheme $\mathcal{C}$ of $\mathrm{Spec}(A)$. Let $\lambda : \Lambda \to \mathrm{Spec}(A)$ denote the blowing-up with center $\mathcal{C}$. Let $\tilde{\mathcal{C}}$ denote the inverse image of $\mathcal{C}$ by $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$, and $\sigma : \Sigma \to \mathrm{Spec}(B)$ denote the blowing-up with center $\tilde{\mathcal{C}}$. We have the commutative diagram below.

$$\begin{tikzcd}
\Sigma \arrow[r, "\sigma"] \arrow[d, "\pi"'] & \mathrm{Spec}(B) \arrow[d] \\
\Lambda \arrow[r, "\lambda"] & \mathrm{Spec}(A)
\end{tikzcd}$$

It is easy to see that for every closed point $r \in \Lambda$ lying over the unique closed point of $\mathrm{Spec}(A)$, there exists a unique closed point $s \in \Sigma$ lying over the unique closed point of $\mathrm{Spec}(B)$ such that $\pi(s) = r$. Thus, for the player $\langle II \rangle$, choosing a closed point $r \in \Lambda$ and choosing a closed point $s \in \Sigma$ are equivalent. Since the player $\langle I \rangle$ can win the game for $f \in A$, $\langle I \rangle$ can win the first framed resolution game for (B,P), $\ell = 1$ and $f_1 = f \in A$.

By Lemma 7.2, for every Weierstrass representation (h,g,t) over (B,P), the player $\langle I \rangle$ can choose a sequence $\{i_\nu \mid 1 \le \nu < \omega + 1\}$ described in Lemma 7.2.1, when he plays the seventh framed resolution game for (h,g,t).

Assume that $\omega < \infty$. Then, for $t_\omega = t + H_1 + H_2 + \cdots + H_\omega$, (h, g, t_ω) is a Weierstrass representation satisfying the Abhyankar condition which is not reductive. This follows from Definition 7.1. The player $\langle I \rangle$ can win the sixth framed resolution game for (h, g, t).

Since $AN(B, P)$ is false, there exists a separable Weierstrass representation (h, g, t) over (B, P) such that $\mathrm{w}(g,\, t) < \infty$ and $\omega = \infty$. By Proposition 7.3 we obtain claims of Theorem 1.1. The separable condition in $(5.\nu)$ follows from Lemma 5.3.4 and Lemma 5.12.1. ☐

References

1. Shreeram S. Abhyankar, *Local Uniformization on Algebraic Surfaces over Ground Fields of Characteristic* $p \neq 0$, Ann. of Math. **63**, no. 3, (1956), 491–525.
2. ———, *Resolution of Singularities of Embedded Algebraic Surfaces, Second Edition*, Berlin: Springer, 1998.
3. Edward Bierstone and Pierre D. Milman, *Semianalytic and Subanalytic Sets*, Publ. Math. IHES, **67**, (1988), 5–42.
4. ———, *Uniformization of Analytic Spaces*, J. Amer. Math. Soc, **2**, (1989), 801–836.
5. Heisuke Hironaka, *Resolution of Singularities of an Algebraic Variety over a Field of Characteristic Zero: I, II*, Ann. of Math. **79**, no. 1, (1964), 109–326.
6. Franz-Viktor Kuhlmann, *On Local Uniformization in Arbitrary Characteristic I, II*, Preprint, http://math.usask.ca/~fvk/index.html, (1997, 1998).
7. Hideyuki Matsumura, *Commutative Algebra, Second Edition*, Reading, Massachusetts: Benjamin/Cummings 1980.
8. Mark Spivakovsky, *Resolution of Singularities*, Preprint, (1996).
9. Tohsuke Urabe, *Resolution of Singularities of Germs in Characteristic Positive associated with Valuation Rings of Iterated Divisor Type*, Preprint, http://xxx.lanl.gov/abs/math.AG/9901048, (1999).
10. Oscar Zariski and Pierre Samuel, *Commutative Algebra*, Vol. I, II, Berlin: Springer, 1975.

DEPARTMENT OF MATHEMATICAL SCIENCES, IBARAKI UNIVERSITY, MITO, IBARAKI, 310-8512, JAPAN
E-mail address: `urabe@mito.ipc.ibaraki.ac.jp`