THE GAUSS MAP AND THE DUAL VARIETY OF REAL-ANALYTIC SUBMANIFOLDS IN A SPHERE OR IN HYPERBOLIC SPACE

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ABSTRACT. We study the Gauss map and the dual variety of a real-analytic immersion of a connected compact real-analytic manifold into a sphere or into hyperbolic space. The dual variety is defined to be the set of all normal directions of the immersion. First, we show that the image of the Gauss map characterizes the manifold. Also we show that the dual variety characterizes the manifold. Besides, duality of the second fundamental form and some results on degeneration are obtained.

1. INTRODUCTION

The study of the behavior of tangent spaces is a basic subject in geometry. In this article we focus our study on real-analytic manifolds in an N-sphere or in hyperbolic N-space. We will show that the image of the Gauss map characterizes the manifold.

By $\epsilon = \pm 1$ we denote the sign to characterize these two cases throughout this article. When we consider an N-sphere, ϵ denotes $+1$, and for hyperbolic N-space $\epsilon = -1$.

Let V denote an $(N+1)$-dimensional real vector space with an inner product $(\quad,\quad)$. We assume that V has a basis $b_0, b_1, \ldots, b_N$ satisfying $(b_0, b_0) = \epsilon$, $(b_i, b_i) = 1$ for $1 \le i \le N$ and $(b_i, b_j) = 0$ for $i \ne j$. By S we denote the standard N-sphere or the standard hyperbolic N-space. In the case of the sphere

$$S = \{a \in V \mid (a, a) = 1\},$$

and in the case of the hyperbolic space

$$S = \text{ one of the two connected components of } \{a \in V \mid (a, a) = -1\}.$$

Let M be an n-dimensional differentiable manifold, and $\sigma : M \to S$ be an immersion. For every point $p \in M$ by $\hat{T}_p(M)$ we denote the linear space spanned by the image $\sigma_* T_p(M) \subset V$ of the tangent space of M at p and the vector $\sigma(p)$. (We have identified $T_{\sigma(p)}(V)$ with V here.) Associating each point $p \in M$ with $\hat{T}_p(M)$, we can define a map $g : M \to G(n+1, V)$ to the Grassmann manifold parameterizing $(n+1)$-dimensional linear subspaces in V. We call the map g the (generalized) *Gauss map*.

We say that $\sigma : M \to S$ is *almost injective*, if there is an open dense subset in M such that σ is injective on it. In the case of the sphere by $\tau : S \to S$ we denote the antipodal map $\tau(a) = -a$. Our first main result is the following:

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Theorem 1.1. *Let M and M' be n-dimensional smooth connected compact real-analytic manifolds. Let $\sigma : M \to S$ and $\sigma' : M' \to S$ be almost injective real-analytic immersions. Assume that the image of the associated Gauss map $M \to G(n+1, V)$ and the image of $M' \to G(n+1, V)$ coincide as sets. Then, there is an real-analytic isomorphism $\phi : M \to M'$ such that either $\sigma = \sigma'\phi$ holds, or S is a sphere and $\sigma = \tau\sigma'\phi$ holds.*

The Gauss map for submanifolds of an N-sphere or of hyperbolic N-space has been studied by many authors. (See Obata [9], Sakaki [11, 12].) However, the above fundamental result does not seem to have been known to differential geometers. On the other hand, among algebraic geometers the theory of dual varieties has been well-known. (See Griffiths, Harris [6], Kleiman [8], Piene [10], Urabe [14], Wallace [15].) According to the theory of dual varieties, one finds that an analogous result to the above Theorem 1.1 holds for algebraic varieties in a complex projective space.

Let $\mathbf{CP}^N = \mathbf{CP}(V \otimes \mathbf{C})$ be the complex projective space associated with the complex vector space $V \otimes \mathbf{C}$. Let X be an algebraic subvariety of $\mathbf{CP}^N$. (Recall that the common zero-point set on $\mathbf{CP}^N$ of finitely many homogeneous polynomials satisfying the irreducible condition is called an algebraic subvariety.) Let X_{smooth} denote the set of smooth points on X. For every point $p \in X_{\text{smooth}}$ the *embedded tangent space* $\tilde{T}_p(X)$ is defined. It is a complex projective linear subspace in $\mathbf{CP}^N$ with the same dimension as X such that p is a singular point of $X_{\text{smooth}} \cap \tilde{T}_p(X)$. A complex projective hyperplane $\tilde{H}$ in $\mathbf{CP}^N$ is called a *tangent hyperplane* to X, if $\tilde{H}$ contains $\tilde{T}_p(X)$ for some $p \in X_{\text{smooth}}$. The set of all tangent hyperplanes to X can be regarded as a set in the dual complex projective space $\mathbf{CP}^{N\vee} = \mathbf{CP}(V^* \otimes \mathbf{C})$ associated with the dual vector space $V^* = \mathrm{Hom}(V, \mathbf{R})$. The closure $X^\vee$ in $\mathbf{CP}^{N\vee}$ of the set of all tangent hyperplanes to X is called the *dual variety* of X. It is not difficult to show that $X^\vee$ is an algebraic subvariety in $\mathbf{CP}^{N\vee}$ and the the second dual variety $(X^\vee)^\vee$ coincides with X.

On the other hand, the correspondence $p \in X_{\text{smooth}} \mapsto \tilde{T}_p(X)$ defines the Gauss map from X_{smooth} to a complex Grassmann manifold parameterizing complex linear subspaces in $V \otimes \mathbf{C}$. By the definition it is obvious that the dual variety $X^\vee$ is determined by the image of the Gauss map. Thus one can conclude that an analogous result to Theorem 1.1 holds for algebraic subvarieties in a complex projective space.

Besides, we can conclude immediately that Theorem 1.1 holds, if we replace the word "real-analytic" by the word "real-algebraic." An immersion $\sigma : M \to S$ of an n-dimensional differentiable manifold M is said to be *real-algebraic*, if there is an algebraic subvariety X of complex dimension n in $\mathbf{CP}(V \otimes \mathbf{C})$ defined by finitely many homogeneous polynomials with coefficients in real numbers such that the intersection $X \cap \mathbf{RP}(V)$ coincides with the image of the map $M \to S \hookrightarrow V - \{0\} \to \mathbf{RP}(V)$, where $\mathbf{RP}(V)$ denotes the real projective space associated with V. We regard $\mathbf{RP}(V)$ as a subset of $\mathbf{CP}(V \otimes \mathbf{C})$ by the natural inclusion $\mathbf{RP}(V) \hookrightarrow \mathbf{CP}(V \otimes \mathbf{C})$.

Note here that it is very hard to check whether a given manifold is real-algebraic or not. Maybe this is the main reason why differential geometers do not have respected the importance of the theory of the dual variety. Therefore the generalization for the real-analytic case or for the C^∞ case is very important

When we try to give the generalization for real-analytic varieties, we encounter several difficulties. Consider the basic situation. Let $\bar{M}$ be a smooth connected compact real-analytic manifold, and $\bar{M} \hookrightarrow \mathbf{RP}(V)$ be an embedding (i.e., an injective immersion). For $\bar{M}$ the embedded tangent space $\tilde{T}_p(\bar{M})$ of $\bar{M}$ at a point $p \in \bar{M}$ is a real projective linear space in $\mathbf{RP}(V)$. It is easy to define the dual variety $\bar{M}^\vee$ in the dual real projective space $\mathbf{RP}(V^*)$ as a set. The set of real projective hyperplanes H in $\mathbf{RP}(V)$ such that there is a point $p \in \bar{M}$ with $\tilde{T}_p(\bar{M}) \subset H$ is defined to be $\bar{M}^\vee$. However, in general $\bar{M}^\vee$ is not a real-analytic variety but a so-called subanalytic set. This is the first difficulty. Besides, therefore, we cannot define the second dual variety $(\bar{M}^\vee)^\vee$.

In this article we apply ideas of Whitney and Thom (Bruhat, Whitney [3], Whitney [16], Thom [13]) to overcome these difficulties. Their idea is sometimes represented by the word "stratification". The origin of their idea seems to be the real-analytic case. However, the fundamental theory of real-analytic sets including stratification theory is not still well-developed, compared with the algebraic case and the complex-analytic case. Despite that the real-analytic case is the most important for application, it is hard to construct the theoretical foundation. The main task in this article is to make their idea down-to-earth in the real-analytic case.

Now, we have explained difficulties in our problem using the projective geometry. We return to our main situation in the beginning part. We consider an immersion $\sigma : M \to S$ of an n-dimensional differential manifold M to a sphere or hyperbolic space, because spheres and hyperbolic spaces would be more important than real projective spaces for differential geometers.

Let $S^\vee = \{a \in V \mid (a,\, a) = 1\}$. Note that $S^\vee = S$ in the case of the sphere, and $S^\vee \cap S = \emptyset$ in the case of the hyperbolic space. We will define the dual variety $M^\vee$ of M as a subset of $S^\vee$. Note here that if a non-zero vector $a \in V$ is orthogonal to $\hat{T}_p(M)$ for some $p \in M$, then $(a,\, a) > 0$. We define

$$M^\vee = \{a \in S^\vee \mid \text{The vector } a \text{ is orthogonal to } \hat{T}_p(M) \text{ for some } p \in M.\}.$$

Theorem 1.2. *Let M and M' be n-dimensional smooth connected compact real-analytic manifolds. Let $\sigma : M \to S$ and $\sigma' : M' \to S$ be almost injective real-analytic immersions. Assume that their dual varieties $M^\vee$ and ${M'}^\vee$ coincide as sets. Then, there is an real-analytic isomorphism $\phi : M \to M'$ such that either $\sigma = \sigma'\phi$ holds, or S is a sphere and $\sigma = \tau\sigma'\phi$ holds.*

Theorem 1.1 follows from Theorem 1.2.

The dual variety $M^\vee$ has many interesting properties. Among them Theorem 4.11 in Section 4 is noteworthy. It shows that the second fundamental form of $M^\vee$ is the dual of the second fundamental form of M. It implies also that the second fundamental form of $M^\vee$ depends on only the second order differentials of M. Despite that the dual variety $M^\vee$ is defined by the first order differentials of M, to describe the second fundamental form of $M^\vee$ one never needs the third order information.

In Section 2 we review some fundamental results on real-analytic varieties and real-analytic maps. Theorem 2.31 is the key in this article. It plays a basic role in geometry of real-analytic manifolds. Besides, we develop also the theory of ordinary singularities. Ordinary singularities are singularities on the image of a proper immersion from a smooth manifold. Section 3 is devoted to the proof for Theorem 1.1 and Theorem 1.2. In Section 4 we study the dual variety and the

Gauss map from the point of view of the differential geometry. Theorem 4.11 is the main achievement. In Section 5 we consider degeneration of the dual variety and degeneration of the Gauss map. Theorem 5.4 is the main result in this section.

In this article we always assume that every manifold has a countable basis of topology.

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2. Real-analytic sets and real-analytic maps

In this section we review fundamental properties of real-analytic sets and real-analytic maps. Perhaps differential geometers are not familiar with the theory developed below. We try to give some details. We will apply Theorem 2.14, Proposition 2.16, Corollary 2.32, and Lemma 2.33 in the next section.

Let Q be a real-analytic manifold.

Lemma 2.1 (Theorem of identity). *Assume that Q is connected. Let f be a real-analytic function over Q. Let $U \subset Q$ be a non-empty open subset. If $f(q) = 0$ for every point $q \in U$, then f is identically zero over Q.*

Let $\mathcal{A}$ denote the *sheaf* on Q of real-analytic functions with values in real numbers. A correspondence $\mathcal{F}$ satisfying some conditions corresponding an arbitrary open subset $U \subset Q$ to an abelian group $\mathcal{F}(U)$ is called a sheaf on Q. For $\mathcal{A}$, $\mathcal{A}(U)$ is the ring of real-analytic functions over U with values in real numbers. For every point $q \in Q$ the stalk $\mathcal{A}_q = \varinjlim_{U \ni q} \mathcal{A}(U)$ is isomorphic to the ring of convergent power series with coefficients in real numbers, and is a Noetherian ring with a unique maximal ideal. We denote the maximal ideal $\{f \in \mathcal{A}_q \mid f(q) = 0\}$ of $\mathcal{A}_q$ by M_q. In the theory below the ring $R = \mathcal{A}(Q)$ of real-analytic functions defined over Q plays an important role. The ring R is not necessarily Noetherian.

The most important property of the sheaf $\mathcal{A}$ is *coherence*.

A sheaf $\mathcal{F}$ on Q of $\mathcal{A}$-modules is said to be *finitely generated*, if for every point $q \in Q$ we have an open neighborhood U of q, a positive integer m and a surjective sheaf morphism $(\mathcal{A}|U)^m \to \mathcal{F}|U$ of $\mathcal{A}$-modules on U, where $\mathcal{A}|U$ and $\mathcal{F}|U$ denote the restriction to U of sheaves $\mathcal{A}$ and $\mathcal{F}$ respectively. A finitely generated sheaf $\mathcal{F}$ of $\mathcal{A}$-modules is said to be *coherent*, if for every open set U, for every positive integer m, and for every sheaf morphism $\phi : (\mathcal{A}|U)^m \to \mathcal{F}|U$ of $\mathcal{A}$-modules, the kernel $\operatorname{Ker}\phi$ is finitely generated.

Let $E \subset R$ be an ideal of the ring R. An ideal sheaf $\tilde{E}$ in $\mathcal{A}$ is defined associated with E. By $E\mathcal{A}_q$ we denote the ideal in the stalk $\mathcal{A}_q$ at $q \in Q$ generated by E. For an open set U we define the corresponding ideal by $\tilde{E}(U) = \{f \in \mathcal{A}(U) \mid f \in E\mathcal{A}_q \text{ for every } q \in U\}$.

Theorem 2.2 (Gunning, Rossi [7, Chapter IV]).

1. *$\mathcal{A}$ is a coherent sheaf of $\mathcal{A}$-modules.*
2. *For every ideal $E \subset R$ the sheaf $\tilde{E}$ is a coherent ideal sheaf of $\mathcal{A}$-modules.*
3. *A finitely generated subsheaf of a coherent sheaf is coherent.*
4. *Let $0 \to \mathcal{F} \to \mathcal{F}' \to \mathcal{F}'' \to 0$ be an exact sequence of sheaves of $\mathcal{A}$-modules. If two of $\mathcal{F}$, $\mathcal{F}$ and $\mathcal{F}''$ are coherent, then the other is also coherent.*

Corollary 2.3. *A sheaf $\mathcal{F}$ of $\mathcal{A}$-modules is coherent if and only if for every point $q \in Q$ we have a neighborhood U of q, positive integers l, m, and an exact sequence $(\mathcal{A}|U)^l \to (\mathcal{A}|U)^m \to \mathcal{F}|U \to 0$ of sheaves of $\mathcal{A}$-modules.*

Proof. The "only if" part follows from the definition of coherence. We show the "if" part. By Theorem 2.2.1 and 4 $(\mathcal{A}|U)^m$ is coherent. In the above exact sequence the kernel of $(\mathcal{A}|U)^m \to \mathcal{F}|U$ is finitely generated, and thus, it is coherent by Theorem 2.2.3. Again by Theorem 2.2.4 we conclude $\mathcal{F}|U$ is coherent. Since $q \in Q$ is arbitrary, we conclude $\mathcal{F}$ is coherent. □

For every sheaf $\mathcal{F}$ on Q we can define the cohomology group $H^m(Q,\ \mathcal{F})$ for every non-negative integer m. (Gunning, Rossi [7], Cartan [4])

Theorem 2.4 (Cartan [4], Grauert [5])**.** *When Q is a real-analytic manifold, for every coherent sheaf $\mathcal{F}$ of $\mathcal{A}$-modules and for every integer $m > 0$ $H^m(Q, \mathcal{F}) = 0$.*

Lemma 2.5. *Let $\mathcal{J}$ be a coherent sheaf of ideals in $\mathcal{A}$. Let $E = \mathcal{J}(Q)$ be the ideal in R of global sections of $\mathcal{J}$. Then, $\mathcal{J} = \tilde{E}$.*

Proof. Let $q \in Q$ be an arbitrarily fixed point.

First, we show $\mathcal{J}_q = E\mathcal{A}_q$. Let $\mathcal{M}$ denote the ideal sheaf in $\mathcal{A}$ defined by the condition $\mathcal{M}(U) = \{f \in \mathcal{A}(U) \mid f(q) = 0\}$ for any open set U with $q \in U$ and $\mathcal{M}(U) = \mathcal{A}(U)$ if $q \notin U$. Since $\mathcal{M}|Q - \{q\} \cong \mathcal{A}|Q - \{q\}$, $\mathcal{M}$ is coherent around any point p with $p \neq q$. Let U_0 be a coordinate neighborhood around q and $z_1, z_2, \ldots, z_n$ be the local coordinates with $z_1(q) = z_2(q) = \cdots = z_n(q) = 0$. Let E_0 be the ideal in $\mathcal{A}(U_0)$ generated by $z_1, z_2, \ldots, z_n$. By definition $\mathcal{M}|U_0 = \tilde{E}_0$. Replacing Q in Theorem 2.2.2 by U_0 one knows that $\mathcal{M}|U_0$ is coherent. Thus $\mathcal{M}$ is coherent. By Theorem 2.2.4 and Corollary 2.3 we can conclude $\mathcal{G} = \mathcal{J} \otimes_{\mathcal{A}} (\mathcal{A}/\mathcal{M})$ is also coherent. Note that by definition $\mathcal{G}_q \cong \mathcal{J}_q \otimes_{\mathcal{A}_q} (\mathcal{A}_q/M_q)$ and $\mathcal{G}_p = 0$ if $p \neq q$. We have a surjective morphism $\mathcal{J} \to \mathcal{G}$ of sheaves of $\mathcal{A}$-modules. The sheaf $\mathcal{F} = \operatorname{Ker}(\mathcal{J} \to \mathcal{G})$ is also coherent. We have an exact sequence $E = \mathcal{J}(Q) \to \mathcal{J}_q \otimes_{\mathcal{A}_q} (\mathcal{A}_q/M_q) \cong \mathcal{G}(Q) \to H^1(Q,\ \mathcal{F})$. By Theorem 2.4 $H^1(Q,\ \mathcal{F}) = 0$. Thus $E \to \mathcal{J}_q \otimes_{\mathcal{A}_q} (\mathcal{A}_q/M_q)$ is surjective, and $\mathcal{J}_q = E\mathcal{A}_q + M_q\mathcal{J}_q$. Since $\mathcal{F}$ is coherent, $\mathcal{F}_q$ is a finitely generated $\mathcal{A}_q$-module. By Nakayama's lemma in commutative algebra we can conclude $\mathcal{J}_q = E\mathcal{A}_q$.

Now, by the definition of $\tilde{E}$ there is an injective sheaf morphism $\mathcal{J} \to \tilde{E}$. By the above and by the definition of $\tilde{E}$ one knows $\mathcal{J}_q = \tilde{E}_q$. Since $q \in Q$ is arbitrary, one concludes $\mathcal{J} = \tilde{E}$. □

A subset $X \subset Q$ is said to be *real-analytic* or *analytic*, if for every point $q \in X$ we have a neighborhood U of q and finitely many real-analytic functions $f_1, f_2, \ldots, f_s$ on U such that $X \cap U = \{p \in U \mid f_1(p) = f_2(p) = \cdots = f_s(p) = 0\}$. Note that the point q runs not over Q but over X. Thus, by definition any analytic subset X is not necessarily closed but *locally closed*, in other words, we have an open set U such that X is a closed subset of U.

For a sheaf $\mathcal{F}$ on Q the set of points $q \in Q$ such that the stalk $\mathcal{F}_q$ at q is non-zero is called the *support* of $\mathcal{F}$, and is denoted by $\operatorname{Supp}(\mathcal{F})$.

Lemma 2.6. *Let $\mathcal{F}$ be a coherent sheaf of $\mathcal{A}$-modules on Q.*

1. *For a point $q \in Q$, $\mathcal{F}_q = 0$ if and only if $\mathcal{F}_q \otimes_{\mathcal{A}_q} (\mathcal{A}_q/M_q) = 0$.*
2. $\operatorname{Supp}(\mathcal{F})$ *is a closed analytic set in Q.*

Proof. 1. By Nakayama's lemma we obtain the claim.
2. Let $p \in Q$ be an arbitrary point. By coherence we have an open neighborhood U of p and an exact sequence $(\mathcal{A}|U)^l \xrightarrow{\phi} (\mathcal{A}|U)^m \to \mathcal{F}|U \to 0$ for some positive integers l, m. The morphism ϕ is represented by a $m \times l$ matrix with entries $\phi_{i,j} \in \mathcal{A}(U)$. For every point $q \in U$ we have an exact sequence $\mathbf{R}^l \xrightarrow{\phi(q)} \mathbf{R}^m \to \mathcal{F}_q \otimes_{\mathcal{A}_q} (\mathcal{A}_q/M_q) \to 0$, where $\phi(q)$ denotes the linear map defined by the matrix $(\phi_{i,j}(q))$ evaluated at q. Let $f_1, f_2, \ldots, f_s \in \mathcal{A}(U)$ be $m \times m$ minors of the matrix $(\phi_{i,j})$. One knows that $\mathcal{F}_q \otimes_{\mathcal{A}_q} (\mathcal{A}_q/M_q) \neq 0$ if and only if $f_i(q) = 0$ for $1 \leq i \leq s$. Thus by 1 we have $\mathrm{Supp}\,(\mathcal{F}) \cap U = \{q \in U \mid f_i(q) = 0 \text{ for } 1 \leq i \leq s\}$. Since $p \in Q$ is arbitrary, we obtain the conclusion. □

For an ideal E in the ring R we denote

$$V(E) = \{q \in Q \mid f(q) = 0 \text{ for every } f \in E.\}.$$

By definition $V(E)$ is a closed subset of Q.

Lemma 2.7. $V(E) = \mathrm{Supp}\,(\mathcal{A}/\tilde{E})$ *for any ideal* E *in the ring* R.

Proof. Let $q \in Q$ be an arbitrary point. By definition $q \in V(E)$ if and only if $E\mathcal{A}_q \subset M_q$. On the other hand, by the definition of the sheaf $\tilde{E}$ we have $\tilde{E}_q = E\mathcal{A}_q$. Thus $q \notin V(E)$ if and only if $\tilde{E}_q = \mathcal{A}_q$. Since $(\mathcal{A}/\tilde{E})_q \cong \mathcal{A}_q/\tilde{E}_q$ for stalks, we can conclude $V(E) = \mathrm{Supp}\,(\mathcal{A}/\tilde{E})$. □

Note that an ideal E in the ring R of analytic functions on Q is not necessarily finitely generated. However, the following proposition holds:

Proposition 2.8. *The set* $V(E)$ *is a closed analytic set for any ideal* E *in the ring* R.

Proof. By Lemma 2.7 $V(E) = \mathrm{Supp}\,(\mathcal{A}/\tilde{E})$. By Theorem 2.2.2 and 4 the ideal sheaf $\tilde{E}$ and the sheaf $\mathcal{A}/\tilde{E}$ are coherent. By Lemma 2.6.2 we conclude that $V(E)$ is closed and analytic. □

An analytic subset $X \subset Q$ is said to be *global* or, more precisely, *global in* Q referring to Q, if we have an ideal E in the ring R such that $X = V(E)$. Any global analytic set in Q is closed in Q.

Proposition 2.9. *A subset* $X \subset Q$ *is a global analytic set if and only if we have a coherent ideal sheaf* $\mathcal{J}$ *in* $\mathcal{A}$ *such that* $X = \mathrm{Supp}\,(\mathcal{A}/\mathcal{J})$.

Proof. Let X be a global analytic set. By Lemma 2.7 we have an ideal $E \subset R$ with $X = \mathrm{Supp}\,(\mathcal{A}/\tilde{E})$. By Theorem 2.2.2 the ideal sheaf $\tilde{E}$ is coherent.

Conversely, assume that $X = \mathrm{Supp}\,(\mathcal{A}/\mathcal{J})$ for some coherent sheaf $\mathcal{J}$ of ideals. By Lemma 2.5 $\mathcal{J} = \tilde{E}$ for $E = \mathcal{J}(Q)$. By Lemma 2.7 we have $X = V(E)$. Thus X is global. □

For any subset $X \subset Q$ the ideal

$$I(X) = \{f \in R \mid f(q) = 0 \text{ for every point } q \in X.\}$$

is called the *ideal of* X. It is uniquely determined by X.

Besides, for $X \subset Q$ we can define the ideal sheaf $\mathcal{I}_X$. For an open set $U \subset Q$ the corresponding ideal is given by

$$\mathcal{I}_X(U) = \{f \in \mathcal{A}(U) \mid f(q) = 0 \text{ for every point } q \in X \cap U.\}.$$

$\mathcal{I}_X$ is called *the ideal sheaf of* X. By definition $I(X) = \mathcal{I}_X(Q)$ and we have the canonical injective sheaf morphism $\widetilde{I(X)} \to \mathcal{I}_X$ of $\mathcal{A}$-modules. Note that in general sheaves $\widetilde{I(X)}$ and $\mathcal{I}_X$ do not necessarily coincide.

Corollary 2.10. *If the ideal sheaf* $\mathcal{I}_X$ *of a closed analytic set* X *is coherent, then* X *is global.*

Proof. It is not difficult to show $X = \mathrm{Supp}\,(\mathcal{A}/\mathcal{I}_X)$, if X is closed and analytic. Thus the corollary follows from Proposition 2.9. □

Remark.
1. In Cartan [4] we find an example of a closed analytic set which is *not* global in the case $Q = \mathbf{R}^3$.
2. Let $Q = \mathbf{R}^3$ and $X = \{(x, y, z) \in \mathbf{R}^3 \mid x^2 = y^2 z\}$. By definition X is a global analytic set. However, the ideal sheaf $\mathcal{I}_X$ is not coherent around the origin.

A global analytic set $X \subset Q$ is said to be *reducible*, if we have global analytic sets X' and X'' satisfying $X = X' \cup X''$, $X \neq X'$ and $X \neq X''$. We say that a global analytic set X is *irreducible*, if X is not reducible.

The lemma below follows from the definitions.

Lemma 2.11.
1. *If* $E \subset E'$ *for ideals* E, E' *in* R, *then* $V(E) \supset V(E')$
2. *If* $X \subset X'$ *for subsets* X, X' *in* Q, *then* $I(X) \supset I(X')$.
3. $X \subset V(I(X))$ *for any subset* $X \subset Q$. *The quality* $X = V(I(X))$ *holds if and only if* X *is a global analytic set.*
4. $I(X \cup X') = I(X) \cap I(X')$ *for any subsets* X, X'. *Moreover, if* X *and* X' *are global analytic sets, then so is the union* $X \cup X'$.
5. *Let* X_λ $(\lambda \in \Lambda)$ *be a family of global analytic sets. Let* E_0 *be the ideal in* R *generated by the union* $\bigcup_{\lambda\in\Lambda} I(X_\lambda)$. *Then* $\bigcap_{\lambda\in\Lambda} X_\lambda = V(E_0)$. *In particular,* $\bigcap_{\lambda\in\Lambda} X_\lambda$ *is also a global analytic set.*
6. *A global analytic set* X *is irreducible if and only if* $I(X)$ *is a prime ideal.*

Lemma 2.12. *Let* X_m $(m = 1, 2, 3, \dots)$ *be a sequence of global analytic sets in* Q. *We assume that* $X_m \supset X_{m+1}$ *for every* m. *Then, for every compact subset* $K \subset Q$, *there is an integer* m_0 *such that* $X_m \cap K = X_{m_0} \cap K$ *for* $m \geq m_0$.

Proof. Let $E_m = I(X_m)$ and $E_\infty = \bigcup_m E_m$. Since $E_m \subset E_{m+1}$ for every m, E_∞ is an ideal in R. By Theorem 2.2.2 $\mathcal{J}_\infty = \tilde{E}_\infty$ and $\mathcal{J}_m = \tilde{E}_m$ are coherent ideal sheaves. Let $q \in K$ be an arbitrary point. We can choose a finitely generated subideal $D_q \subset E_\infty$ with $(\mathcal{J}_\infty)_q = E_\infty \mathcal{A}_q = D_q \mathcal{A}_q$, since $(\mathcal{J}_\infty)_q$ is a finitely generated $\mathcal{A}_q$-module. Since D_q is finitely generated, there is a number m_q with $D_q \subset E_{m_q}$. This implies $(\mathcal{J}_\infty)_q = (\mathcal{J}_{m_q})_q$. Thus $(\mathcal{J}_\infty/\mathcal{J}_{m_q})_q = 0$. Since $\mathcal{J}_\infty/\mathcal{J}_{m_q}$ is coherent, there is an open neighborhood U_q of q in Q such that $(\mathcal{J}_\infty/\mathcal{J}_{m_q})|U_q = 0$. We have $\mathcal{J}_{m_q}|U_q = \mathcal{J}_\infty|U_q$. Since there are injective morphisms $\mathcal{J}_{m_q} \hookrightarrow \mathcal{J}_m \hookrightarrow \mathcal{J}_\infty$ for $m \geq m_q$, one concludes that $\mathcal{J}_m|U_q = \mathcal{J}_\infty|U_q$ for every $m \geq m_q$. Since K is compact, we have finitely many points $q_1, q_2, \dots, q_s \in K$ with $K \subset U = \bigcup_{i=1}^s U_{q_i}$. Set $m_0 = \max_{1\leq i\leq s} m_{q_i}$. We have $\mathcal{J}_m|U = \mathcal{J}_\infty|U$ for every $m \geq m_0$. By Lemma 2.7 we have $X_m \cap U = V(E_m) \cap U = \mathrm{Supp}\,((\mathcal{A}/\mathcal{J}_m)|U) = \mathrm{Supp}\,((\mathcal{A}/\mathcal{J}_\infty)|U)$ for $m \geq m_0$. Since $X_m \cap K = (X_m \cap U) \cap K$, we obtain the claim. □

For subsets X, U in Q, we write

$$X_U = V(I(X \cap U)).$$

Lemma 2.13.
1. X_U *is a global analytic set.*

2. *If Y is a global analytic set, and if $Y \supset X \cap U$, then $Y \supset X_U$.*
3. *If X is a global analytic set, then $X \cap U = X_U \cap U$.*
4. $X_U = (X_U)_U$. $(X \cup X')_U = X_U \cup X'_U$.

Theorem 2.14 (Bruhat, Cartan [2]). *Let K be a compact subset of Q. For every global analytic set X in Q there exists a unique finite family of global analytic sets $\{Y_1, Y_2, \dots, Y_s\}$ satisfying the following conditions:*

1. $X_K = Y_1 \cup Y_2 \cup \cdots \cup Y_s$
2. *Y_i is irreducible for every i.*
3. *$Y_i \not\subset Y_j$ for any $i \neq j$.*

Proof. First, we will show the existence of the finite family $\{Y_1, Y_2, \dots, Y_s\}$ satisfying the condition 1 for X and the condition 2. Let Γ denote the set of global analytic sets X such that any finite family $\{Y_1, Y_2, \dots, Y_s\}$ does not satisfy the condition 1 for X or the condition 2. Assume $\Gamma \neq \emptyset$. We will deduce a contradiction. Choose an element $X \in \Gamma$. If X_K is irreducible, then for $s = 1$ the set $\{X_K\}$ satisfies the two conditions. Thus X_K is reducible and we can write $X_K = X' \cup X''$ for some global analytic sets X', X'' with $X_K \neq X'$ and $X_K \neq X''$. If $X' \notin \Gamma$ and $X'' \notin \Gamma$, then we have $X \notin \Gamma$, a contradiction, because $X_K = (X' \cup X'')_K = X'_K \cup X''_K$. Thus we can assume $X' \notin \Gamma$, after exchanging X' and X'', if necessary. Since $X \supset X_K \supset X'$, we have $X \cap K \supset X' \cap K$. If $X \cap K = X' \cap K$, then we have $X_K \subset X'$, and $X_K = X'$, a contradiction. Thus $X \cap K \neq X' \cap K$. Repeating similar arguments, one sees that there is a sequence of elements $X_m \in \Gamma$ $(m = 1, 2, 3, \dots)$ with $X_1 = X$, $X_2 = X'$ such that $X_m \supset X_{m+1}$ and $X_m \cap K \neq X_{m+1} \cap K$ for every m. This contradicts Lemma 2.12. Thus $\Gamma = \emptyset$.

Let $\Delta = \{Y_1, Y_2, \dots, Y_s\}$ be a finite family satisfying the condition 1 for X and the condition 2. If we have numbers i and j such that $i \neq j$ and $Y_i \subset Y_j$, also the set $\Delta' = \Delta - \{Y_i\}$ satisfies the condition 1 for X and the condition 2. Thus we can replace Δ by Δ'. Repeating this procedure in finite steps, we can obtain the set satisfying all three conditions.

We will next show the uniqueness. Let $\Delta = \{Y_1, Y_2, \dots, Y_s\}$ and $\Delta' = \{Y'_1, Y'_2, \dots, Y'_t\}$ be two families satisfying the above three conditions for X. Since $X_K = Y_1 \cup Y_2 \cup \cdots \cup Y_s = Y'_1 \cup Y'_2 \cup \cdots \cup Y'_t$, we have $Y_i = \bigcup_{j=1}^t (Y_i \cap Y'_j)$ for every i. Since Y_i is irreducible and the right-hand side is a global analytic set, one knows that for each i there is a number j with $Y_i = Y_i \cap Y'_j$. The last condition is equivalent to $Y_i \subset Y'_j$. Similarly, one knows that for each j there is a number k with $Y'_j \subset Y_k$. Thus, for each i we have numbers j, k with $Y_i \subset Y'_j \subset Y_k$. By the condition 3 we have $i = k$ and $Y_i = Y'_j$. This implies $\Delta \subset \Delta'$. Similarly we have $\Delta \supset \Delta'$, and thus $\Delta = \Delta'$. □

Consider Theorem 2.14 when X is compact. We can choose a compact set K with $X \subset K$, and then we have $X_K = X$. In this case the equality in the condition 1 is called the *irreducible decomposition* of X, and $Y_1, Y_2, \dots, Y_s$ are called the *irreducible components* of X.

To develop local theory the concept of germs is important.

Fix a point $q \in Q$ and consider a pair (U, X), where $U \subset Q$ is a neighborhood of q and X is a subset of U. We say that two pairs (U, X) and (U', X') is *equivalent*, if we have a neighborhood U'' such that $q \in U'' \subset U \cap U'$ and $X \cap U'' = X' \cap U''$. An equivalence class is called a *germ* or a germ of sets at q, and the equivalence class represented by (U, X) is denoted by (X, q). If (X, q) has a representative (U, X)

such that X is an analytic set, we say that (X, q) is a germ of analytic sets, and if X is a closed analytic set, we say that it is a germ of closed analytic sets.

Let (X, q) and (X', q) be two germs represented by (U, X) and (U', X') respectively. Let $U'' = U \cap U'$. The germ represented by $(U'', (X \cap U'') \cup (X' \cap U'')$ depends only on (X, q) and (X', q) and does not depend on the representatives. We denote it by $(X, q) \cup (X', q)$, and call it the union of germs (X, q) and (X', q). Similarly the germ represented by $(U'', (X \cap U'') \cap (X' \cap U''))$ is denoted by $(X, q) \cap (X', q)$ and called the intersection of (X, q) and (X', q). Besides, if we have a neighborhood U_1 of q such that $U_1 \subset U \cap U'$ and $X \cap U_1 \subset X' \cap U_1$, we denote $(X, q) \subset (X', q)$ or $(X', q) \supset (X, q)$. For any finite number of germs at q we can define the union and the intersection of them.

We would like to consider the irreducible decomposition of a germ of closed analytic sets.

Let (X, q) be a germ of closed analytic sets. We say that (X, q) is *reducible*, if we have germs (X', q) and (X'', q) of closed analytic sets satisfying $(X, q) = (X', q) \cup (X'', q)$, $(X, q) \neq (X', q)$ and $(X, q) \neq (X'', q)$. If (X, q) is not reducible, we say that it is *irreducible*.

Proposition 2.15. *Let $q \in Q$ be a point. For every germ (X, q) of closed analytic sets at q there exists a unique finite family $\{(Y_1, q), (Y_2, q), \ldots, (Y_s, q)\}$ of germs of closed analytic sets at q satisfying the following conditions:*

1. $(X, q) = (Y_1, q) \cup (Y_2, q) \cup \cdots \cup (Y_s, q)$
2. (Y_i, q) *is irreducible for every* i.
3. $(Y_i, q) \not\subset (Y_j, q)$ *for any* $i \neq j$.

The equality in the condition 1 is called the *local irreducible decomposition* of a germ (X, q), and $(Y_1, q), (Y_2, q), \ldots, (Y_s, q)$ are called the *local irreducible components* of (X, q).

For the proof of Proposition 2.15 we need describe the relation between ideals in the local ring $\mathcal{A}_q$ at q and germs at q of closed analytic sets.

Let E be an ideal in the local ring $\mathcal{A}_q$ at q. Since $\mathcal{A}_q$ is Noetherian, we have a finite number of generators $f_1, f_2, \ldots, f_s$ of E. Let U be a neighborhood of q such that f_i's are all convergent on it. Let $X = \{p \in U \mid f_i(p) = 0 \text{ for } 1 \leq i \leq s\}$. It is easily checked that the germ of closed analytic sets represented by (U, X) depends only on the ideal E and does not depend on the choice of generators. We denote it by $V(E)$.

Conversely, consider a germ (X, q) of closed analytic sets represented by (U, X), where X is a closed analytic set in a neighborhood U of q. We can associate an ideal $I(X, q)$ with (X, q). The ideal $I(X, q)$ is a collection of elements $f \in \mathcal{A}_q$ such that we have a neighborhood U' of q contained in U such that f is convergent on U' and $f(p) = 0$ for every point $p \in X \cap U'$.

We do not give the proof of Proposition 2.15, because it is very similar to that of Theorem 2.14 and is easier.

Let $X \subset Q$ be an analytic set, and $q \in X$ be a point. We say that X is *smooth* at q, if there is a coordinate neighborhood U of q, local analytic coordinates $z_1, z_2, \ldots, z_n$ on U with the origin q, and a non-negative integer m such that $X \cap U = \{p \in U \mid z_i(p) = 0 \text{ for } m < i \leq n\}$. If X is smooth at q, then the number m does not depend on the choice of U and $z_1, z_2, \ldots, z_n$. We call the number m the *dimension* of X at q. If X is not smooth at q, then we say that X is *singular* at q, or q is a singular point of X. We say that X is smooth, if X has no singular point.

Let (X, q) be a germ of closed analytic sets at $q \in Q$. We say that (X, q) is smooth, if it has a representative (U, X) such that X is a closed analytic set in U which is smooth at q. We say that (X, q) has an *ordinary singularity*, if every local irreducible components at q is smooth.

An analytic subset $X \subset Q$ is said to *have only ordinary singularities*, if for every point $q \in X$ the germ (X, q) has an ordinary singularity. Let $X \subset Q$ be an analytic set with only ordinary singularities. A point $q \in X$ is singular, if and only if (X, q) has two or more local irreducible components. The set of singular points $\operatorname{Sing} X$ is an analytic set contained in X. The singular set $\operatorname{Sing} X$ is closed in X and $X - \operatorname{Sing} X$ is dense in X by Lemma 2.1.

Proposition 2.16. *Let $X \subset Q$ be an analytic subset with only ordinary singularities. We say that an analytic map $f : P \to Q$ is X-admissible, if P is a smooth analytic manifold, the image $f(P)$ is contained in X and for every connected component P_0 of P, $f(P_0) \not\subset \operatorname{Sing} X$.*

1. *There exists a smooth analytic manifold $\hat{X}$ and an X-admissible analytic map $\nu : \hat{X} \to Q$ such that for any X-admissible analytic map $f : P \to Q$ there is a unique analytic map $\hat{f} : P \to \hat{X}$ with $f = \nu \hat{f}$.*
2. *If there is another smooth analytic manifold $\hat{X}'$ and an X-admissible analytic map $\nu' : \hat{X}' \to Q$ satisfying the same conditions as in 1, then there is an isomorphism $\rho : \hat{X} \to \hat{X}'$ of real-analytic manifolds with $\nu = \nu' \rho$.*
3. *The map $\nu : \hat{X} \to Q$ is an immersion with $\nu(\hat{X}) = X$ and the induced map $\hat{X} \to X$ by ν is proper. The set $\hat{X} - \nu^{-1}(\operatorname{Sing} X)$ is open and dense in $\hat{X}$ and ν is injective on $\hat{X} - \nu^{-1}(\operatorname{Sing} X)$.*
4. *If a proper X-admissible map $f : P \to Q$ satisfies $f(P) = X$, then the induced map $\hat{f} : P \to \hat{X}$ is surjective.*

The map $\nu : \hat{X} \to Q$ is called the *normalization* of X. It is unique up to isomorphisms.

Proof. 1. Let $\hat{X}$ be the set of pairs $(q, (Y, q))$ where q is a point in X and (Y, q) is a local irreducible component of X at q, and $\nu : \hat{X} \to Q$ be the map $\nu(q, (Y, q)) = q$.

We will define the structure of a smooth analytic manifold on $\hat{X}$. First, we introduce the topology on $\hat{X}$. Let $\hat{q} = (q, (Y, q))$ be a point on $\hat{X}$. By definition we have a neighborhood U of q in Q, and finitely many closed smooth analytic sets $Y_1, Y_2, \ldots, Y_s$ in U such that $X \cap U = Y_1 \cup Y_2 \cup \cdots \cup Y_s$ and each Y_i contains the point q. We can assume $(Y_1, q) = (Y, q)$. A neighborhood of $\hat{q}$ in $\hat{X}$ is defined to be a set in the form $\{(p, (Y_1, p)) \mid p \in Y_1 \cap U'\}$ where U' is a neighborhood of q with $U' \subset U$. It is easily checked that these neighborhoods define topology on $\hat{X}$. By definition any neighborhood is homeomorphic to a smooth analytic set with the form $Y_1 \cap U'$. Thus the structure of a smooth analytic manifold on $\hat{X}$ is defined. We can check that ν is analytic and X-admissible.

Second, we will show that $\nu : \hat{X} \to Q$ has the property described above. Consider an X-admissible analytic map $f : P \to Q$. We claim that for every $p \in P$ there is a unique local irreducible component $(Y, f(p))$ of $(X, f(p))$ such that for some neighborhood W of p and for some representative (U, Y) of $(Y, f(p))$ the inclusion relation $f(W) \subset Y$ holds.

Fix a point $p \in P$ and put $q = f(p)$. We have a neighborhood U of q and closed smooth analytic sets $Y_1, Y_2, \ldots, Y_s$ as above. Put $Z_i = f^{-1}(Y_i)$ for $1 \leq i \leq s$. Since the germ (P, p) is irreducible, $(Z_i, p) = (P, p)$ for some i. Assume that we have numbers i, j such that $i \neq j$ and $(Z_i, p) = (Z_j, p) = (P, p)$. We have a neighborhood W of p in P with $W \subset Z_i \cap Z_j$. Thus $f(W) \subset Y_i \cap Y_j \subset \operatorname{Sing} X$. Since $\operatorname{Sing} X$ is an analytic set which is closed in X, and since $f(P) \subset X$, the inverse image $f^{-1}(\operatorname{Sing} X)$ is a closed analytic subset in P. Let P_0 be the connected component of P containing the point p. We have $P_0 \subset f^{-1}(\operatorname{Sing} X)$ by Lemma 2.1, since $W \subset f^{-1}(\operatorname{Sing} X)$. It contradicts that f is X-admissible. We conclude that there is a unique number i with $(Z_i, p) = (P, p)$. We obtain the desired claim.

We fix a neighborhood W of p, the local irreducible component $(Y, f(p))$ of $(X, f(p))$, and a representative (U, Y) of $(Y, f(p))$ such that $f(W) \subset Y$. We define the map $\hat{f} : P \to \hat{X}$ by $\hat{f}(p) = (f(p), (Y, f(p)))$. By definition and by the above claim we have $\hat{f}(p') = (f(p'), (Y, f(p')))$ for any $p' \in W$. Let $\hat{Y} = \{(q', (Y, q')) \mid q' \in Y\}$. It is a neighborhood of $\hat{f}(p)$ in $\hat{X}$. The map ν induces an isomorphism $\bar{\nu} : \hat{Y} \to Y$. Since $f(W) \subset Y$, we can define the composition $\bar{\nu}^{-1} f|W$ and we have $\hat{f}|W = \bar{\nu}^{-1} f|W$. Thus $\hat{f}$ is analytic on W. Since $p \in P$ is arbitrary, $\hat{f} : P \to \hat{X}$ is an analytic map. By definition it satisfies $\nu \hat{f} = f$.

Assume that we have another analytic map $\hat{f}' : P \to \hat{X}$ with $\nu \hat{f}' = f$. By the above claim we have $\hat{f}|W = \bar{\nu}^{-1} f|W = \hat{f}'|W$. Since $p \in P$ is arbitrary, we have $\hat{f} = \hat{f}'$.

2. By the condition in 1 we have a map $\rho' : \hat{X}' \to \hat{X}$ with $\nu\rho' = \nu'$. By the same condition for $\hat{X}'$ we have a map $\rho : \hat{X} \to \hat{X}'$ with $\nu'\rho = \nu$. Thus $\nu' = \nu'\rho\rho'$ and $\nu = \nu\rho'\rho$. By the uniqueness in the condition in 1 we conclude that $\rho\rho'$ and $\rho'\rho$ are the identity maps.

3. It is obvious by the definition of $\hat{X}$ above.

4. Since f and the induced map $\hat{X} \to X$ by ν are proper, and since $f = \nu\hat{f}$, one sees that also $\hat{f}$ is proper. In particular, the image $\hat{f}(P) \subset \hat{X}$ is closed. On the other hand by 3, $\hat{f}(P)$ contains a dense subset $\hat{X} - \nu^{-1}(\operatorname{Sing} X)$. Thus $\hat{f}(P) = \hat{X}$. □

Proposition 2.17. *Any closed analytic subset $X \subset Q$ with only ordinary singularities is global in Q.*

Proof. Let $q \in Q$ be a point. By Corollary 2.10 we have only to show that the ideal sheaf $\mathcal{I}_X$ is coherent around q.

If $q \notin X$, then $\mathcal{I}_X$ is isomorphic to $\mathcal{A}$ around q and is coherent around q by Theorem 2.2.1.

Assume $q \in X$. By assumption we have a neighborhood U of q in Q, and a finite number of closed smooth analytic sets $Y_1, Y_2, \ldots, Y_s$ in U passing through q such that $X \cap U = Y_1 \cup Y_2 \cup \cdots \cup Y_s$.

First, we see that $\mathcal{I}_{Y_i}$ is coherent for every i. We can assume that there are local analytic coordinates $z_1, z_2, \ldots, z_n$ on U with the origin q such that $Y_i = \{p \in U \mid z_j(p) = 0 \text{ for } m < j \leq n\}$ where m is the dimension of Y_i. Let E be the ideal generated by z_j $(m < j \leq n)$ in $\mathcal{A}(U)$. By definition $\tilde{E} = \mathcal{I}_{Y_i}$. Thus by Theorem 2.2.2 we can conclude that $\mathcal{I}_{Y_i}$ is coherent.

Put $X_i = Y_1 \cup Y_2 \cup \cdots \cup Y_i$. By induction on i we show that $\mathcal{I}_{X_i}$ is coherent. If $i = 1$, $\mathcal{I}_{X_1} = \mathcal{I}_{Y_1}$ is coherent. Assume $i > 1$. Since $\mathcal{I}_{X_i} = \mathcal{I}_{X_{i-1}} \cap \mathcal{I}_{Y_i}$, we have an exact sequence

$$0 \to \mathcal{I}_{X_i} \to \mathcal{I}_{X_{i-1}} \oplus \mathcal{I}_{Y_i} \xrightarrow{\alpha} \mathcal{A}|U.$$

By Theorem 2.2.4 and by induction assumption $\mathcal{I}_{X_{i-1}} \oplus \mathcal{I}_{Y_i}$ is coherent. Let $\mathcal{F}$ be the image of α. Since $\mathcal{F}$ is the image of a coherent sheaf, it is finitely generated. Since $\mathcal{F}$ is a subsheaf of $\mathcal{A}|U$, it is coherent by Theorem 2.2.3. By Theorem 2.2.4 $\mathcal{I}_{X_i}$ is coherent.

Since $\mathcal{I}_{X_s} = \mathcal{I}_X|U$, $\mathcal{I}_X$ is coherent around q. □

By Theorem 2.14 we have the following corollary:

Corollary 2.18. *Any compact analytic subset $X \subset Q$ with only ordinary singularities has the irreducible decomposition.*

The *rank* $\mathrm{rank}_q(F)$ of a subset F in R at a point $q \in Q$ is the maximal number of linearly independent differentials $(df_1)_q, (df_1)_q, \ldots, (df_s)_q$ at q where f_i's are elements in F. Obviously by definition $\mathrm{rank}_q(F)$ is not greater than the dimension of Q at q. The *rank* $\mathrm{rank}_q(X)$ of a point set $X \subset Q$ at $q \in X$ is $\mathrm{rank}_q(I(X))$. The *rank* $\mathrm{rank}(X)$ of X is the largest value of $\mathrm{rank}_q(I(X))$ for $q \in X$. We do not define $\mathrm{rank}(X)$ when X is an empty set.

Lemma 2.19. 1. *Let E be an ideal in R, and F be a subset generating E. Then, $\mathrm{rank}_q(F) = \mathrm{rank}_q(E)$ at every point $q \in V(E)$.*
2. *For every integer m the set $\{q \in Q \mid \mathrm{rank}_q(F) \le m\}$ is a global analytic set in Q.*

Proof. 1. Since $F \subset E$ we have $\mathrm{rank}_q(F) \le \mathrm{rank}_q(E)$. For every $f \in E$ there are finite elements $g_1, g_2, \ldots, g_s \in F$ and $h_1, h_2, \ldots, h_s \in R$ such that $f = \sum_{i=1}^s h_i g_i$. Then, we have $df_q = \sum_{i=1}^s h_i(q)(dg_i)_q$, since $g_i(q) = 0$ for $q \in V(E)$. Thus the linear subspace spanned by df_q $(f \in E)$ is contained in the linear subspace spanned by dg_q $(g \in F)$. This implies $\mathrm{rank}_q(F) \ge \mathrm{rank}_q(E)$.
2. Let $U \subset Q$ be a coordinate neighborhood and $z_1, z_2, \ldots, z_n$ be the local coordinates on U. Let E_U be the ideal in $\mathcal{A}(U)$ generated by $m \times m$ minors of the matrix $\partial f/\partial z_i$ $(f \in F, 1 \le i \le n)$. We have $\{q \in Q \mid \mathrm{rank}_q(F) \le m\} \cap U = V(E_U) = \mathrm{Supp}((\mathcal{A}|U)/\tilde{E}_U)$. By Theorem 2.2.2 $\tilde{E}_U$ is a coherent sheaf of ideals on U. On the other hand, we can check that the ideal E_U does not depend on the choice of the local coordinates $z_1, z_2, \ldots, z_n$. Thus for any two coordinate neighborhoods U, U' we have $\tilde{E}_U|U \cap U' = \tilde{E}_{U'}|U \cap U'$. One knows that there is a coherent sheaf of ideals $\mathcal{J}$ in $\mathcal{A}$ such that $\mathcal{J}|U = \tilde{E}_U$ for every coordinate neighborhood U. By Proposition 2.9 $\{q \in Q \mid \mathrm{rank}_q(F) \le m\} = \mathrm{Supp}(\mathcal{A}/\mathcal{J})$ is global. □

Theorem 2.20 (Whitney [16])**.** *Let Q be a connected real-analytic manifold of dimension n. Assume that there is a set $F = \{f_1, f_2, \ldots, f_n\}$ of n global analytic functions on Q such that $\mathrm{rank}_q(F) = n$ for every point $q \in Q$. Let X be a global analytic set in Q. Any point $q \in X$ with $\mathrm{rank}_q(X) = \mathrm{rank}(X)$ is a smooth point of X with dimension equal to $n - \mathrm{rank}(X)$.*

Proof. By assumption $f_1, f_2, \ldots, f_n$ are local coordinates around every point $q \in Q$. We can define the partial derivative $\partial g/\partial f_i \in R$ for any $g \in R$ and for $1 \le i \le n$. Let $k = \mathrm{rank}(X)$. Choose and fix a point $q \in X$ with $\mathrm{rank}_q(X) = k$. By definition we have elements $g_1, g_2, \ldots, g_k \in I(X)$ with $\mathrm{rank}_q(\{g_1, g_2, \ldots, g_k\}) = k$.

Let $Y = \{p \in Q \mid g_1(p) = g_2(p) = \cdots = g_k(p) = 0\}$. Since $g_1, g_2, \ldots, g_k \in I(X)$, we have $X \subset Y$.

There are $(n-k)$ elements $f_{i_1}, f_{i_2}, \ldots, f_{i_{n-k}}$ in F such that $\mathrm{rank}_q(\{g_1, g_2, \ldots, g_k, f_{i_1}, f_{i_2}, \ldots, f_{i_{n-k}}\}) = n$. We denote $g_{k+j} = f_{i_j}$ for $1 \le j \le n-k$. Let $Z = \{p \in$

$Q \mid \mathrm{rank}_p(\{g_1, g_2, \ldots, g_n\}) < n\}$. By Lemma 2.19.2 Z is a global analytic set with $q \notin Z$. Let (J_{ij}) be the adjoint matrix of the Jacobian matrix $(\partial g_i/\partial f_j)$ and $J = \det(\partial g_i/\partial f_j)$ be the Jacobian determinant. J_{ij} and J are elements of R. We have $Z = \{p \in Q \mid J(p) = 0\}$ and $J(q) \neq 0$.

For any $h \in \mathcal{A}(Q - Z)$ we can define the partial derivative $\partial h/\partial g_i \in \mathcal{A}(Q - Z)$ for $1 \leq i \leq n$, and $\partial h/\partial g_i = \left(\sum_{j=1}^n J_{ij}\partial h/\partial f_j\right)/J$. In particular, if $h \in R = \mathcal{A}(Q)$ then $J(\partial h/\partial g_i) \in R$ for $1 \leq i \leq n$.

We here claim that if $h \in I(X)$ then $J^2(\partial h/\partial g_i) \in I(X)$ for $k < i \leq n$. Indeed, by the definition of k $\mathrm{rank}_p(\{g_1, g_2, \ldots, g_k, h\}) \leq k$ for any point $p \in X - Z$. The rank in the left-hand side of this inequality is equal to the rank of the matrix with the i-th row $(\partial g_i/\partial g_j)(p)$ $(1 \leq j \leq n)$ for $1 \leq i \leq k$ and the $(k+1)$-th row $(\partial h/\partial g_j)(p)$ $(1 \leq j \leq n)$. Since $(\partial g_i/\partial g_j)(p) = 0$ for $1 \leq i \leq k$, $1 \leq j \leq n$ with $i \neq j$, and since $(\partial g_i/\partial g_i)(p) = 1$ for $1 \leq i \leq k$ we conclude that $(\partial h/\partial g_j)(p) = 0$ for $k < j \leq n$ and for $p \in X - Z$. Since $J(\partial h/\partial g_i) \in R$, we have $(J^2(\partial h/\partial g_i))(p) = 0$ for any $p \in X \cup Z$. Thus we obtain the desired claim.

Let $H = \{h \in \mathcal{A}(Q - Z) \mid J^m h \in I(X)$ for some positive integer $m\}$. This R-module H has three properties. First, $I(X) \subset H$. Second, $h(p) = 0$ for any $h \in H$ and for any $p \in X - Z$. If $h \in H$, then $J^m h \in I(X)$ for some positive integer m. Thus $J(p)^m h(p) = 0$ for $p \in X - Z$. Since $J(p) \neq 0$ for $p \in X - Z$, we have $h(p) = 0$. Third, if $h \in H$ and if $k < i \leq n$, then $\partial h/\partial g_i \in H$. Now, by assumption we have a positive integer m with $J^m h \in I(X)$. By the above claim $J^2(\partial J^m h/\partial g_i) = m(J^m h)J(\partial J/\partial g_i) + J^{m+2}(\partial h/\partial g_i) \in I(X)$. Since $J^m h \in I(X)$ and $J(\partial J/\partial g_i) \in R$, we have $m(J^m h)J(\partial J/\partial g_i) \in I(X)$. Thus $J^{m+2}(\partial h/\partial g_i) \in I(X)$ and $\partial h/\partial g_i \in H$.

The three properties above imply that the value at q of any higher derivatives with respect to $g_{k+1}, g_{k+2}, \ldots, g_n$ of any $h \in I(X)$ vanishes. Let $U \subset Q$ be an open neighborhood of q such that $g_1, g_2, \ldots, g_n$ are the local coordinates on U. One knows that for any $h \in I(X)$ there are $h_1, h_2, \ldots, h_k \in \mathcal{A}(U)$ such that $h = \sum_{i=1}^k h_i g_i$. Thus $h(p) = 0$ for any $h \in I(X)$ and for any $p \in Y \cap U$. One concludes $Y \cap U \subset X = V(I(X))$ and thus $Y \cap U = X \cap U$. The set $Y \cap U$ is a smooth analytic set of dimension $n - k$ and we conclude the proof. □

Corollary 2.21. *Assume the same condition for Q as in Theorem 2.20. Any global analytic subset $X \subset Q$ has a global analytic subset X_1 such that $X_1 \subset X$, $X_1 \neq X$ and $X - X_1$ is smooth.*

Proof. Let $X_1 = \{q \in Q \mid \mathrm{rank}_q(X) < \mathrm{rank}(X)\} \cap X$. By the definition of $\mathrm{rank}(X)$ one knows $X_1 \neq X$. By Lemma 2.19.2 X_1 is global. By Theorem 2.20 $X - X_1$ is smooth. □

Corollary 2.22. *The set of smooth points of any analytic set $X \subset Q$ is open and dense in X.*

Proof. Any point $q \in X$ has a neighborhood U such that $X \cap U$ is global in U and there are local coordinates $z_1, z_2, \cdots, z_n$ on U. For any connected open set U' with $q \in U' \subset U$ the pair $(U', X \cap U')$ satisfies the assumption in Theorem 2.20. One can conclude that $X \cap U'$ contains a smooth point. Thus the set of smooth points is dense in X. By definition of a smooth point it is open in X. □

Corollary 2.23. *Any analytic set $X \subset Q$ is a union of countably many smooth connected analytic sets.*

Proof. First, we show that for any point $q \in X$ we have a neighborhood U of q in Q, a finite sequence $X_1, X_2, \dots, X_s$ of global analytic sets in U such that $X_1 = X \cap U$, $X_s = \emptyset$, $X_m \supset X_{m+1}$ for $1 \leq m < s$ and $X_m - X_{m+1}$ is a smooth analytic set for $1 \leq m < s$.

Let W be a coordinate neighborhood of $q \in X$ in Q such that $X \cap W$ is global in W. Let U be a neighborhood of $q \in X$ in Q contained in W such that the closure $K = \bar{U}$ is compact and is contained in W. We denote $Y_1 = X \cap W$ and put $Z_1 = (Y_1)_K$. Z_1 is the minimal global analytic set in W with $Y_1 \cap K = Z_1 \cap K$. Applying Corollary 2.21 to Z_1 we have a global analytic set Y_2 in W such that $Z_1 \supset Y_2$, $Z_1 \neq Y_2$ and $Z_1 - Y_2$ is smooth. If $Z_1 \cap K = Y_2 \cap K$, then we have $Z_1 = (Z_1)_K = (Y_2)_K \subset Y_2$ and thus $Z_1 = Y_2$, a contradiction. Thus $Z_1 \cap K \neq Y_2 \cap K$. Put $Z_2 = (Y_2)_K$. Since $Z_2 \subset Y_2$ and $Z_2 \cap K = Y_2 \cap K$, we have $Z_1 \supset Z_2$, $Z_1 \cap K \neq Z_2 \cap K$ and $(Z_1 - Z_2) \cap U$ is smooth.

If Z_2 is not empty, we repeat the same procedure replacing Y_1 by Y_2.

Assume that the repetition of this procedure does not terminate in finite steps. Then, we have an infinite sequence Z_m $(m = 1, 2, 3, \dots)$ of global analytic sets in W such that $Z_m \supset Z_{m+1}$ and $Z_m \cap K \neq Z_{m+1} \cap K$ for every m. This contradicts Lemma 2.12.

Thus we have a finite sequence $Z_1, Z_2, \dots, Z_s$ of global analytic sets in W such that $Z_1 = (X \cap W)_K$, $Z_s = \emptyset$, $Z_m \supset Z_{m+1}$ and $(Z_m - Z_{m+1}) \cap U$ is smooth for $1 \leq m < s$. Put $X_m = Z_m \cap U$ for $1 \leq m \leq s$. Since $X_1 = X \cap U$, X_m's satisfy the desired conditions.

Note that since Q has a countable basis of topology, $X_m - X_{m+1}$ has at most countably many connected components for each m. Thus one knows that each point $q \in X$ has a neighborhood U_q such that $X \cap U_q$ is a union of countably many smooth connected analytic sets. Since Q has a countable basis of topology, we have countably many points $q_m \in X$ $(m = 1, 2, 3 \dots)$ with $X \subset \bigcup_m U_{q_m}$. Thus we can conclude the proof. □

Remark. By a result in Bruhat, Cartan [1] we can show that the number of connected components of $X_m - X_{m+1}$ is finite in the above proof. Thus any compact analytic set $X \subset Q$ is a union of finite number of smooth connected analytic sets.

The largest value of the dimension of a non-empty analytic set X at a smooth point $q \in X$ is called the *dimension* of X, and is denoted by $\dim X$. We define that the dimension of the empty set is equal to $-\infty$. For a germ (Y, q) of analytic sets the smallest value of $\dim Y$ of a representative (U, Y) for (Y, q) is called the *dimension* of (Y, q). For a given analytic set $X \subset Q$ the dimension of the germ (X, q) at a point $q \in X$ is called the dimension of X at q, and is denoted by $\dim_p X$. If q is a smooth point of X, the definition of $\dim_p X$ here coincides with the previous definition.

Lemma 2.24. 1. *Let X, Y be analytic subsets in Q with $X \subset Y$. If Y is smooth, then* $\dim X \leq \dim Y$.

2. *If* $\dim X < \dim Q$ *for an analytic subset X of a connected real-analytic manifold Q, then X has Lebesque measure zero in Q.*

Proof. 1. Easy.

2. By Corollary 2.23 we have countably many smooth analytic sets X_m $(m = 1, 2, 3, \dots)$ with $X = \bigcup_m X_m$. By the definition of the dimension at a smooth point we have $\dim X_m \leq \dim Q$ for every m. If $\dim X_m = \dim Q$ for some m, then

by Lemma 2.1 X_m and X contain an open set of Q and we have $\dim X = \dim Q$, a contradiction. Thus $\dim X_m < \dim Q$ for every m. It is not difficult to show that the Lebesque measure of any smooth analytic subset of dimension less than n is zero. Thus X_m has Lebesque measure zero for every m. By σ-additivity of Lebesque measure we conclude that X has Lebesque measure zero. □

Remark. We guess that the following five claims hold:

1. Admitting countably many irreducible components, every global analytic set has the irreducible decomposition.
2. $\dim(X) + \mathrm{rank}\,(X) = \dim(Q)$ for any non-empty global analytic set X in Q.
3. If X is a global irreducible analytic set in Q, then $\dim(Y) < \dim(X)$ for any global analytic set Y in Q with $Y \subset X$ and $Y \neq X$.
4. Theorem 2.20 holds without the assumption on the existence of a set F.
5. Claim 1 in Lemma 2.24.1 holds without assumption that Y is smooth.

However, we do not try to show them in this article, because we will not apply them later in this article and they are not the main subjects of this article. Perhaps, introducing the concept of the Krull dimension in commutative ring theory, we would be able to solve some of them.

We would like to consider a real-analytic map below.

Let P be another real-analytic manifold, and $f : P \to Q$ be a real-analytic map. We denote $\mathcal{A}_P$ or $\mathcal{A}_Q$ instead of $\mathcal{A}$, when we need distinguish the sheaf of functions on P from the one of Q.

The *rank* $\mathrm{rank}_p\,(f)$ of a map f at a point $p \in P$ is the rank of the Jacobian matrix of f at p. The *rank* $\mathrm{rank}\,(f)$ of f is the largest value of $\mathrm{rank}_p\,(f)$ for $p \in P$.

Lemma 2.25. *Assume that P is connected and of dimension m. Let X be a global analytic set in P and $X_0 = \{p \in X \mid \mathrm{rank}_p\,(X) = \mathrm{rank}\,(X)\}$. Assume that every point $p \in X_0$ is a smooth point of X with dimension equal to $m - \mathrm{rank}\,(X)$. For every integer k we have a global analytic set Y such that for any point $p \in X_0$, $\mathrm{rank}_p\,(f|X_0) \le k$ if and only if $p \in Y$.*

Remark. Note in the above lemma that we can define the rank of the restricted map $f|X_0 : X_0 \to Q$ at $p \in X_0$, since X_0 is smooth.

Proof. Let $p \in P$ be an arbitrary point. Let U be a coordinate neighborhood of p and $z_1, z_2, \ldots, z_m$ be the local coordinates on U. Let W be a coordinate neighborhood of $f(p)$ in Q with $f(U) \subset W$ and $w_1, w_2, \ldots, w_n$ be the local coordinates on W. We denote $f_j = w_j \circ f$ for $1 \le j \le n$, which is a function on U. Let M_U be a matrix with infinite rows and m columns such that any row of M_U is either $(\partial f_j/\partial z_1, \partial f_j/\partial z_2, \ldots, \partial f_j/\partial z_m)$ for some j with $1 \le j \le n$ or $(\partial g/\partial z_1, \partial g/\partial z_2, \ldots, \partial g/\partial z_m)$ with $g \in I(X)$. It is easy to see that if $p \in X_0$, then $\mathrm{rank}_p\,(f|X_0) = k$ if and only if the matrix M_U evaluated at p has rank $\mathit{rank}\,(X)+k$.

Let E_U be the ideal in $\mathcal{A}_P(U)$ generated by $(\mathit{rank}\,(X) + k) \times (\mathit{rank}\,(X) + k)$ minors of the matrix M_U. By Theorem 2.2.2 $\tilde{E}_U$ is a coherent sheaf of ideals on U. On the other hand, we can check that the ideal E_U does not depend on the choice of the local coordinates $z_1, z_2, \ldots, z_m$ and $w_1, w_2, \ldots, w_n$. Thus for any two coordinate neighborhoods U, U' we have $\tilde{E}_U|U \cap U' = \tilde{E}_{U'}|U \cap U'$. One knows that there is a coherent sheaf of ideals $\mathcal{J}$ in $\mathcal{A}$ such that $\mathcal{J}|U = \tilde{E}_U$ for every coordinate neighborhood U. By Proposition 2.9 $Y = \mathrm{Supp}\,(\mathcal{A}/\mathcal{J})$ is global. By the above remark Y has the desired property. □

Corollary 2.26. *For every integer k the set $\{p \in P \mid \operatorname{rank}_p(f) \leq k\}$ is a global analytic set in P.*

Corollary 2.27. *Assume that P is connected, has dimension m and there is a set $F = \{f_1, f_2, \dots, f_m\}$ of m global analytic functions on P such that $\operatorname{rank}_p(F) = m$ for every point $p \in P$. Any global analytic subset $X \subset P$ has a global analytic subset X_1 such that $X_1 \subset X$, $X_1 \neq X$, $X - X_1$ is smooth, and $\operatorname{rank}_p(f|X - X_1)$ does not depend on $p \in X - X_1$*

Corollary 2.28. *Any analytic set $X \subset P$ is a union of countably many smooth connected analytic sets X_k such that $\operatorname{rank}_p(f|X_k)$ does not depend on $p \in X_k$.*

Proposition 2.29. *Assume that $\operatorname{rank}(f) = \operatorname{rank}_p(f)$ for every $p \in P$ for a real-analytic map $f : P \to Q$. Then, for every point $p \in P$ and for every neighborhood U' of p there is an open set U with $p \in U \subset U'$ such that the image $f(U)$ is a connected smooth analytic subset in Q with dimension equal to $\operatorname{rank}(f)$. The image $f(P)$ is a union of countably many smooth connected analytic sets with dimension equal to $\operatorname{rank}(f)$.*

If, moreover, f is proper, then $f(P)$ is an analytic set with only ordinary singularities such that any local irreducible component of $(f(P), q)$ at any point $q \in f(P)$ has dimension equal to $\operatorname{rank}(f)$.

Proof. Let $k = \operatorname{rank}(f)$. Let $p \in P$ be an arbitrary point. By assumption the Jacobian matrix of f has constant rank k around p. By inverse mapping theorem we can choose local coordinates $z_1, z_2, \dots, z_m$ on a connected neighborhood U_p of p and local coordinates $w_1, w_2, \dots, w_n$ on a neighborhood W_p of $f(p)$ in Q such that $w_i \circ f = z_i$ for $1 \leq i \leq k$ and $w_i \circ f = 0$ for $k < i \leq n$. We denote $X_p = \{q \in W_p \mid w_i(q) = 0 \text{ for } k < i \leq n\}$. Obviously X_p is a smooth connected analytic set with dimension k. We have $f(U_p) \subset X_p$ and $f(U_p)$ is open in X_p. Thus replacing W_p with a smaller one, we can assume $f(U_p) = X_p$. Since U_p is connected, X_p is also connected. Since P has a countable basis of topology, we have countably many points p_l $(l = 1, 2, 3, \dots)$ with $P = \bigcup_l U_{p_l}$. We have $f(P) = \bigcup_l X_{p_l}$.

We here assume moreover that f is proper. We have only to show that $f(P)$ is an analytic set in Q. Let $q \in f(P)$ be an arbitrary point. By the above reasoning one knows that for each point $p \in f^{-1}(q)$ we have a neighborhood U_p of p and a neighborhood W_p of q such that $f(U_p)$ is a closed analytic set in W_p. Since f is proper, the inverse image $f^{-1}(q)$ is compact. We have finite points $p_1, p_2, \dots, p_s \in f^{-1}(q)$ with $f^{-1}(q) \subset \bigcup_{i=1}^s U_{p_i}$. Let $Y = P - \bigcup_{i=1}^s U_{p_i}$. By definition Y is a closed set with $Y \cap f^{-1}(q) = \emptyset$. Thus the image $f(Y)$ is a closed set with $q \notin f(Y)$, since f is proper. We have a neighborhood W of q in Q with $W \cap f(Y) = \emptyset$. Put $W' = W \cap W_{p_1} \cap W_{p_2} \cap \cdots \cap W_{p_s}$ and $U' = f^{-1}(W')$. U' is a neighborhood of $f^{-1}(q)$ in P with $U' = \bigcup_{i=1}^s (U_{p_i} \cap U')$ We have $f(U') = \bigcup_{i=1}^s (f(U_{p_i}) \cap W')$ and we conclude that $f(U') = f(f^{-1}(W'))$ is a closed analytic set in W'. Since $q \in f(P)$ was an arbitrary point, the set $f(P)$ is analytic. □

Corollary 2.30. *Let $X \subset P$ be an analytic set and $r = \max_{p \in X} \operatorname{rank}_p(f)$. The image $f(X)$ of X by f is a union of countably many smooth connected analytic sets with dimension less than or equal to r.*

Proof. By Corollary 2.28 we have smooth connected analytic sets X_k $(k = 1, 2, 3, \dots)$ such that $\operatorname{rank}_p(f|X_k)$ does not depend on $p \in X_k$ and $X = \bigcup_k X_k$. By Proposition 2.29 $f(X_k)$ is a union of countably many smooth connected analytic sets with dimension equal to rank $(f|X_k)$. For a point $p \in X_k$ we have rank $(f|X_k) = \text{rank}_p(f|X_k) \leq \text{rank}_p(f) \leq r$. Thus $f(X_k)$ is a union of countably many smooth connected analytic sets with dimension $\leq r$. Since $f(X) = \bigcup_k f(X_k)$, we have the claim. □

The set $C = \{p \in P \mid \text{rank}_p(f) < \text{rank}(f)\}$ is called the *critical set* of f.

Theorem 2.31 (Generalization of Sard's Theorem for analytic maps)**.**
Let C be the critical set of an analytic map $f : P \to Q$ of real-analytic manifolds. Assume that C does not contain any connected component of P. Then, the set $P - f^{-1}(f(C))$ is dense in P.

Remark. The above theorem does not hold in the category of C^∞-manifolds and C^∞-maps. Let $P = \{(x, y, z) \in \mathbf{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ be the unit 2-sphere and $Q = \mathbf{R}^3$. Let $\phi : \mathbf{R} \to \mathbf{R}$ be a function of class C^∞ such that $\phi(z) = 0$ for $|z| \leq 1/3$, $\phi(z) > 0$ for $|z| > 1/3$, and $\phi(z) = 1$ for $|z| \geq 2/3$. Consider a map $f : P \to Q$ defined by $f(x, y, z) = (\phi(z)x, \phi(z)y, z)$ for $(x, y, z) \in P$. It is easy to see that rank $(f) = 2$, $C = \{p \in P \mid \text{rank}_p(f) < 2\} = \{(x, y, z) \in P \mid |z| \leq 1/3\}$, $f(C) = \{(x, y, z) \in \mathbf{R}^3 \mid x = y = 0, |z| \leq 1/3\}$ and $f^{-1}(f(C)) = C$. In this example $P - f^{-1}(f(C)) = P - C$ is not dense in P.

Proof. First, note that by Lemma 2.1 $P - C$ is dense in P under the assumption, since C is a closed analytic subset in P. Assume that $P - f^{-1}(f(C))$ is not dense in $f(P)$. We will deduce a contradiction. By assumption we have a non-empty open set U' with $U' \subset f^{-1}(f(C))$. We have $(P - C) \cap U' \neq \emptyset$. Pick a point $p_0 \in U' - C$. Since $\text{rank}_p(f) = \text{rank}(f)$ for any $p \in P - C$, by Proposition 2.29 we have a neighborhood U of p_0 such that $p_0 \in U \subset U' - C$ and $f(U)$ is a connected smooth analytic subset in Q with dimension equal to rank (f). Since $U \subset U' \subset f^{-1}(f(C))$, we have $f(U) \subset f(C)$.

On the other hand, by Corollary 2.30 we have a countable family of connected smooth subset X_k $(k = 1, 2, 3, \dots)$ with $\dim X_k < \text{rank}(f)$ such that $f(C) = \bigcup_k X_k$. Putting $Y_k = X_k \cap f(U)$, we have $f(U) = \bigcup_k Y_k$. Note that Y_k is an analytic set since it is the intersection of two analytic sets. By Lemma 2.24.1 $\dim Y_k \leq \dim X_k < \text{rank}(f) = \dim f(U)$. Thus by Lemma 2.24.2 Y_k has measure zero with respect to Lebesque measure on $f(U)$. Since $f(U)$ is the countable union of Y_k's, we conclude that also $f(U)$ has measure zero with respect to Lebesque measure on $f(U)$, which is a contradiction. □

Corollary 2.32. *Let C be the critical set of a proper analytic map $f : P \to Q$ of real-analytic manifolds. Assume that C does not contain any connected component of P. Then, the set $f(P) - f(C)$ is dense in $f(P)$, and is an analytic set with only ordinary singularities. Any local irreducible component of $f(P) - f(C)$ at any point $q \in f(P) - f(C)$ has dimension equal to* rank (f). *The set L_0 of smooth points of $f(P) - f(C)$ is dense in $f(P)$, and the inverse image $f^{-1}(L_0)$ is dense in P.*

Lemma 2.33. *Let $X \subset Q$ be a closed analytic set with only ordinary singularities. Let $\nu : \hat{X} \to Q$ denote the normalization of X.*

1. *X is compact if and only if $\hat{X}$ is compact.*
2. *X is irreducible if and only if $\hat{X}$ is connected.*

Proof. 1. Obvious by Proposition 2.16.3.
2. By Proposition 2.16.3 the map $\nu : \hat{X} \to Q$ is a proper immersion, $W = \hat{X} - \nu^{-1}(\text{Sing}\, X)$ is an open dense subset, and $\nu|W$ is injective.

Assume that $\hat{X}$ is not connected. Let $\hat{X}'$ be a connected component of $\hat{X}$ and $\hat{X}'' = \hat{X} - \hat{X}'$. We have $W \cap \hat{X}' \neq \emptyset$ and $W \cap \hat{X}'' \neq \emptyset$. By Proposition 2.29 $X' = \nu(\hat{X}')$ and $X'' = \nu(\hat{X}')$ are closed analytic subsets of X with only ordinary singularities and by Proposition 2.17 they are global. They satisfy $X = X' \cup X''$. On the other hand, since $\nu|W$ is injective, $X' - \operatorname{Sing} X \neq \emptyset$, $X'' - \operatorname{Sing} X \neq \emptyset$ and $(X' - \operatorname{Sing} X) \cap (X'' - \operatorname{Sing} X) = \emptyset$. Thus $X \neq X'$ and $X \neq X''$. One knows that X is reducible.

Conversely assume that $\hat{X}$ is connected. Let X' and X'' be closed global analytic subsets with $X = X' \cup X''$. We have $\hat{X} = \hat{X}' \cup \hat{X}''$ for $\hat{X}' = \nu^{-1}(X')$ and $\hat{X}'' = \nu^{-1}(X'')$. $\hat{X}'$ and $\hat{X}''$ are closed analytic subsets in $\hat{X}$. If $\hat{X}' = \hat{X}''$, then we have $\hat{X} = \hat{X}' = \hat{X}''$ and $X = X' = X''$. Consider the case $\hat{X}' \not\subset \hat{X}''$. In this case $\hat{X}'$ contains a non-empty open set $\hat{X} - \hat{X}''$. Since $\hat{X}$ is connected and $\hat{X}'$ is a closed analytic subset, we have $\hat{X} = \hat{X}'$. Thus $X = X'$. Similarly in the case $\hat{X}' \not\supset \hat{X}''$ we have $X = X''$. Therefore X is irreducible. □

3. Duality

Recall that V denotes an $(N+1)$-dimensional real vector space with the inner product $(\ ,\ )$. It has a basis $b_0, b_1, \ldots, b_N$ satisfying $(b_0, b_0) = \epsilon = \pm 1$, $(b_i, b_i) = 1$ for $1 \leq i \leq N$ and $(b_i, b_j) = 0$ for $0 \leq i, j \leq N, i \neq j$. By S we denote the standard N-sphere in V when $\epsilon = +1$, and we denote the standard hyperbolic N-space in V when $\epsilon = -1$.

The lemma below follows from Proposition 2.16, Corollary 2.32 and Lemma 2.33.

Lemma 3.1. *Let (M, σ) be a pair of an n-dimensional smooth connected compact real-analytic manifold M and an almost injective real-analytic immersion $\sigma : M \to S$. The image $\sigma(M)$ is an n-dimensional compact irreducible analytic subset with only ordinary singularities in S. The correspondence $(M, \sigma) \mapsto \sigma(M)$ defines a one-to-one correspondence between the set of isomorphism classes of such pairs (M, σ) and the set of n-dimensional compact irreducible analytic subsets with only ordinary singularities in S.*

Below in this section we consider an almost injective real-analytic immersion $\sigma : M \to S$ from an n-dimensional smooth connected compact real-analytic manifold M. In the case $n = N$ any immersion $\sigma : M \to S$ is an isomorphism since S is simply connected and M is compact. Thus it is trivial that in this case our Theorem 1.1 and Theorem 1.2 hold. Also the case $M = \emptyset$ is trivial. We assume $n < N$ and $M \neq \emptyset$ below.

For a subset $Z \subset V$ by $Z^\perp = \{a \in V \mid (a, b) = 0 \text{ for every } b \in Z\}$ we denote the orthogonal complement of Z in V.

Recall that we denoted $S^\vee = \{a \in V \mid (a, a) = 1\}$, and $\hat{T}_p(M)$ was the linear subspace of V spanned by the embedded tangent space $\sigma_* T_p(M)$ of M at p and the vector $\sigma(p)$. We define a manifold $\mathcal{E}$, maps μ and λ as follows:

$$\begin{aligned} \mathcal{E} &= \{(p, a) \in M \times S^\vee \mid a \in S^\vee \cap \hat{T}_p(M)^\perp\} \\ &= \text{The set of pairs } (p, a) \in M \times V \text{ of a point } p \text{ of } M \\ &\qquad \text{and a unit normal vector } a \text{ of } M \text{ at } p \text{ in } S. \end{aligned}$$

$$\mu : \mathcal{E} \to M, \qquad \mu(p, a) = p$$
$$\lambda : \mathcal{E} \to S^\vee, \qquad \lambda(p, a) = a$$

By definition $\mathcal{E}$ is a smooth compact real-analytic manifold, and μ and λ are real-analytic maps. They play very important roles below. The map $\mu : \mathcal{E} \to M$ is the projection of a fiber bundle whose fiber over $p \in M$ is the set of normal unit vectors at p of M in S. The fiber over $p \in M$ is isomorphic to an $(N-n-1)$-sphere. In particular, one knows $\dim \mathcal{E} = N-1$. If $n \leq N-2$, then an $(N-n-1)$-sphere is connected and $\mathcal{E}$ is also connected. The image $\lambda(\mathcal{E}) \subset S^\vee$ coincides with the dual variety $M^\vee$ of M.

We have the diagram below.

$$V \supset S^\vee \xleftarrow{\lambda} \mathcal{E} \xrightarrow{\mu} M \xrightarrow{\sigma} S \subset V$$

Lemma 3.2. 1. *For every point $a \in S^\vee$ the restriction of μ to $\lambda^{-1}(a)$ is an isomorphism.*
2. *For every point $p \in M$ the restriction of λ to $\mu^{-1}(p)$ is an isomorphism.*

Proof. 1. By definition $\lambda^{-1}(a) = \mu(\lambda^{-1}(a)) \times \{a\}$. Thus the first projection μ induces an isomorphism.
2. Similar. □

Note here that by Lemma 3.1 $M^\vee$ is uniquely determined by the analytic set $\bar{M} = \sigma(M)$. We can call $M^\vee$ the dual variety of $\bar{M}$ and can denote $M^\vee = \bar{M}^\vee$.

Let $C \subset \mathcal{E}$ denote the critical set of the map $\lambda : \mathcal{E} \to S^\vee$. We would like to apply Corollary 2.32 to λ. Note that $\mathcal{E}$ is compact and λ is proper. If $\mathcal{E}$ is connected, then obviously $C \not\supset \mathcal{E}$. Assume that $\mathcal{E}$ is not connected. In this case $n = N-1$ and $\mathcal{E}$ has two connected components $\mathcal{E}_+$ and $\mathcal{E}_-$, each of which is isomorphic to M. Multiplying (-1) on the fibers, we have an isomorphism $\tilde{\tau} : \mathcal{E} \to \mathcal{E}$ exchanging $\mathcal{E}_+$ and $\mathcal{E}_-$ and satisfying $\lambda\tilde{\tau} = \tau\lambda$, where $\tau : S^\vee \to S^\vee$ denotes the antipodal map $\tau(a) = -a$. Thus $C_+ = C \cap \mathcal{E}_+$ and $C_- = C \cap \mathcal{E}_-$ coincides with the critical set of the restriction $\lambda|\mathcal{E}_+$ of λ to $\mathcal{E}_+$ and with one of the restriction $\lambda|\mathcal{E}_-$ respectively. Thus $C \not\supset \mathcal{E}_+$ and $C \not\supset \mathcal{E}_-$. By Corollary 2.32 one knows that $\bar{L} = \lambda(\mathcal{E}) - \lambda(C)$ is a real-analytic set with only ordinary singularities, and the set L_0 of smooth points in $\bar{L}$ is dense in $M^\vee$. Also the inverse image $\lambda^{-1}(L_0)$ is dense in $\mathcal{E}$.

We say that a subset $X \subset V$ is *quasi-analytic*, if X contains a smooth analytic subset which is open and dense in X. We have the following:

Proposition 3.3. *The dual variety of a compact irreducible analytic subset with only ordinary singularities in S is a quasi-analytic subset in $S^\vee$.*

Let Y be a smooth analytic set contained in $S^\vee$. For every point $a \in Y$ by $\hat{T}_a(Y)$ we denote the linear space spanned by the tangent space $T_a(Y) \subset V$ of Y at a and the vector a.

We can define the dual variety $X^\vee$ of a quasi-analytic set $X \subset S^\vee$. Let $Y \subset X$ be an open dense smooth analytic subset. We define

$$X^\vee = \text{the closure of } \{a \in S \mid \text{The vector } a \text{ is orthogonal to } \hat{T}_b(Y) \text{ for some } b \in Y.\}.$$

The dual variety $X^\vee$ does not depend on the choice of Y, and is a subset of S.

Theorem 3.4. *Let $\bar{M} \subset S$ be a compact irreducible analytic subset with only ordinary singularities.*

In the case $\epsilon = +1$ of an N-sphere we have $(\bar{M}^\vee)^\vee = \bar{M} \cup \tau(\bar{M})$, where $\tau : S \to S$ denotes the antipodal map.

In the case $\epsilon = -1$ of hyperbolic N-space we have $(\bar{M}^\vee)^\vee = \bar{M}$

Theorem 1.2 follows from Theorem 3.4. Below in this section we show Theorem 3.4.

Before the verification, we explain our notations.

Let W and W' be vector spaces over $\mathbf{R}$. The set of linear maps from W to W' is denoted by $\mathrm{Hom}(W, W')$, which is a vector space over $\mathbf{R}$. The vector space $W^* = \mathrm{Hom}(W, \mathbf{R})$ is called the *dual vector space* of W. The canonical paring $W^* \times W \to \mathbf{R}$ is denoted by $\langle\ \ ,\ \ \rangle$. Thus $\langle \gamma, w\rangle = \gamma(w)$ for $\gamma \in W^* = \mathrm{Hom}(W, \mathbf{R})$ and $w \in W$.

Let $\phi : Q \to W$ be a map from a manifold Q to a vector space W of finite dimension. For every point $q \in Q$ the induced map $\phi_* : T_q(Q) \to T_{\phi(q)}(W)$ between tangent spaces is defined. Identifying $T_{\phi(q)}(W)$ with W, we have a map $\phi_* : T_q(Q) \to W$. The image of the tangent vector $X \in T_q(Q)$ is denoted by $\phi_* X$ and the image of this map is denoted by $\phi_* T_q(Q)$. Note that $\phi_* T_q(Q)$ is not an affine subspace passing the point $\phi(q)$, but a vector subspace passing the origin of W.

In the case that ϕ is an immersion, $\phi_* : T_q(Q) \to W$ is injective. In this case we can identify $T_q(Q)$ and $\phi_* T_q(Q)$ by ϕ_*, and can write $T_q(Q)$ instead of $\phi_* T_q(Q)$.

Let θ be a differential 1-form on Q. For a point $q \in Q$ by θ_q we denote the element in the dual tangent space $T_q^*(Q)$ defined by θ. The real number $\langle \theta_q, X\rangle$ is defined for every tangent vector $X \in T_q(Q)$.

Let $\mathcal{F}$ be the *space of orthogonal normal frames on M*. This space $\mathcal{F}$ consists of points $(p, a_0, a_1, \dots, a_N) \in M \times V^{N+1}$ such that $a_0 = \sigma(p)$, $(a_i, a_i) = 1$ for $1 \le i \le N$, $(a_i, a_j) = 0$ for $i \ne j$ and $a_1, a_2, \dots, a_n$ are a basis of $T_p(M)$. (Note that $(a_0, a_0) = \epsilon$.) It is easy to see that $\mathcal{F}$ is a smooth connected compact real-analytic manifold. We can define a map $\rho : \mathcal{F} \to M$ and maps $f_i : \mathcal{F} \to V$ $(0 \le i \le N)$ by $\rho(p, a_0, a_1, \dots, a_N) = p$ and $f_i(p, a_0, a_1, \dots, a_N) = a_i$. By definition $f_0 = \sigma\rho$ and $\rho : \mathcal{F} \to M$ is the projection of a principal $O(n) \times O(N-n)$-bundle. The group $O(n) \times O(N-n)$ has a right action on the space $\mathcal{F}$. By $R_g : \mathcal{F} \to \mathcal{F}$ we denote the isomorphism induced by $g \in O(n) \times O(N-n)$.

Let $q \in \mathcal{F}$ and $p = \rho(q) \in M$. By definition $f_0(q) = \sigma(p)$, n of vectors $f_1(q), f_2(q), \dots, f_n(q)$ are an orthogonal normal basis of the embedded tangent space $T_p(M)$ of M at p, and $f_{n+1}(q), f_{n+2}(q), \dots, f_N(q)$ are an orthogonal normal basis of the embedded normal space of M at p in S.

A map $\pi : \mathcal{F} \to \mathcal{E}$ is defined by $\pi(q) = (\rho(q), f_N(q))$, since $f_N(q)$ is a unit normal vector of M at $\rho(q) \in M$ in S. This map π is the projection of a principal $O(n) \times O(N-n-1)$-bundle. We have $\mu\pi = \rho$ and $\lambda\pi = f_N$.

We have the commutative diagram below.

$$\begin{array}{ccccccc}
\mathcal{F} & = & \mathcal{F} & = & \mathcal{F} & = & \mathcal{F} \\
\downarrow f_N & & \downarrow \pi & & \downarrow \rho & & \downarrow f_0 \\
V \supset S^{\vee} & \xleftarrow{\ \lambda\ } & \mathcal{E} & \xrightarrow{\ \mu\ } & M & \xrightarrow{\ \sigma\ } & S \subset V
\end{array}$$

Besides, recall that in the case $\epsilon = +1$, $\tau : S \to S$ denotes the antipodal map.

Let $f : \mathcal{F} \to \mathrm{Hom}(\mathbf{R}^{N+1}, V)$ be the map such that for $q \in \mathcal{F}$, the element $f(q) \in \mathrm{Hom}(\mathbf{R}^{N+1}, V)$ is the linear map satisfying $f(q)(x) = \sum_{i=0}^{N} x_i f_i(q)$ for $x = (x_0, x_1, \dots, x_N) \in \mathbf{R}^{N+1}$. By associating the transposed map ${}^t(f(q)) : V^* \to (\mathbf{R}^{N+1})^*$ of $f(q)$ between the dual linear spaces with $q \in \mathcal{F}$, we have a map ${}^t f : \mathcal{F} \to \mathrm{Hom}(V^*, (\mathbf{R}^{N+1})^*)$ with ${}^t f(q) = {}^t(f(q))$. Since f_i's are an orthogonal normal basis of V, we have also the inverse map $(f(q))^{-1} : V \to \mathbf{R}^{N+1}$. We have a map $f^{-1} : \mathcal{F} \to \mathrm{Hom}(V, \mathbf{R}^{N+1})$ with $f^{-1}(q) = (f(q))^{-1}$.

An element $x = (x_0, x_1, \dots, x_N) \in \mathbf{R}^{N+1}$ can be identified with the linear map $\mathbf{R}^{N+1} \to \mathbf{R}$ with $(y_0, y_1, \dots, y_N) \in \mathbf{R}^{N+1} \mapsto \sum_{i=0}^{N} x_i y_i \in \mathbf{R}$. Under the this identification $\mathbf{R}^{N+1} = (\mathbf{R}^{N+1})^*$ and the identification $V = V^*$ induced by the inner product $(\ ,\ )$ we have $If^{-1} = {}^t f$, where I denotes the linear map defined by the diagonal $(N+1) \times (N+1)$-matrix (I_{ij}) whose diagonal entries I_{ii} are $I_{00} = \epsilon$, and $I_{ii} = +1$ for $1 \le i \le N$.

Below we identify the map I and the matrix (I_{ij}) and use this matrix I to keep consistent description in two cases $\epsilon = \pm 1$. When $\epsilon = +1$, you can simply ignore the matrix I.

Let $b_0, b_1, \dots, b_N$ be a basis of V as in the beginning of this section. The dual basis $b_0^*, b_1^*, \dots, b_N^*$ of V^* is defined by the conditions $\langle b_i^*, b_i \rangle = 1$ $(0 \le i \le n)$ and $\langle b_i^*, b_j \rangle = 0$ $(i \ne j)$. The space $\mathbf{R}^{N+1}$ has the standard basis. Let F be the matrix of $f(q) \in \mathrm{Hom}(\mathbf{R}^{N+1}, V)$ associated with the above bases. We can check that ${}^t F I F = I$ where ${}^t F$ denotes the transposed matrix of F. Thus ${}^t F = I F^{-1} I$. Since the isomorphism $L \to L^*$ is represented by the matrix I, one can check that $If^{-1} = {}^t f$ holds.

The differential 1-form ω on $\mathcal{F}$ with values in $\mathrm{Hom}(\mathbf{R}^{N+1}, \mathbf{R}^{N+1})$ is defined by

$$\omega = If^{-1} df = {}^t f df.$$

Note that df is a $\mathrm{Hom}(\mathbf{R}^{N+1}, V)$-valued 1-form on $\mathcal{F}$, and f^{-1} is a $\mathrm{Hom}(V, \mathbf{R}^{N+1})$-valued function. Thus the composition $f^{-1}df$ can be regarded as a $\mathrm{Hom}(\mathbf{R}^{N+1}, \mathbf{R}^{N+1})$-valued 1-form. This form $\omega = (\omega_{ij})$ $(0 \le i, j \le N)$ plays a very important role below. The form ω and its matrix entries ω_{ij} are called the *Maurer-Cartan forms.*

The following lemma is easily checked, since $d(f^{-1}) = -f^{-1}\, df\, f^{-1}$.

Lemma 3.5. 1. *$d\omega + \omega \wedge I\omega = 0$.*
2. *${}^t\omega = -\omega$.*
3. *For every i, j $(0 \le i, j \le N)$, for every point $q \in \mathcal{F}$ and for every tangent vector $X \in T_q(\mathcal{F})$*
$$\langle (\omega_{ij})_q, X \rangle = (f_i(q), (f_j)_* X).$$
4. *$\omega_{i0} = 0$ for $n+1 \le i \le N$.*

Let $\sigma^\vee : L \to S^\vee$ denote the normalization of the analytic set $\bar{L} = \lambda(\mathcal{E}) - \lambda(C)$ with only ordinary singularities, where $C \subset \mathcal{E}$ denotes the critical set of the map $\lambda : \mathcal{E} \to S^\vee$. The map $\sigma^\vee$ is an immersion, the induced map $L \to \bar{L}$ is proper, and $\dim L = \mathrm{rank}\,(\lambda)$.

Denote $\mathcal{E}' = \mathcal{E} - \lambda^{-1}(\lambda(C))$ for simplicity.

Lemma 3.6. *The restriction of λ to $\mathcal{E}'$ is $\bar{L}$-admissible.*

Proof. Let $\mathcal{E}_0$ be a connected component of $\mathcal{E}'$, and $\tilde{p} \in \mathcal{E}_0$ be a point. Since the rank of λ is constant on $\mathcal{E}_0$ and is equal to $\mathrm{rank}\,(\lambda)$, $\tilde{p}$ has a neighborhood U in $\mathcal{E}_0$ such that the image $\lambda(U)$ is a connected smooth analytic set of dimension equal to $\mathrm{rank}\,(\lambda) = \dim \bar{L}$. This implies that the germ $(\lambda(U), \lambda(\tilde{p}))$ is an irreducible component of the germ $(\bar{L}, \lambda(\tilde{p}))$. On the other hand, the germ $(\mathrm{Sing}\,\bar{L}, \lambda(\tilde{p}))$ does not contain any irreducible component of $(\bar{L}, \lambda(\tilde{p}))$. One sees $(\lambda(U), \lambda(\tilde{p})) \not\subset (\mathrm{Sing}\,\bar{L}, \lambda(\tilde{p}))$ and $\lambda(U) \not\subset \mathrm{Sing}\,\bar{L}$. Since $\lambda(U) \subset \lambda(\mathcal{E}_0)$, we have $\lambda(\mathcal{E}_0) \not\subset \mathrm{Sing}\,\bar{L}$. □

By Proposition 2.16 and Lemma 3.6 we have a map $\mu^\vee : \mathcal{E}' \to L$ with $\sigma^\vee \mu^\vee = \lambda|\mathcal{E}'$. Since $\lambda : \mathcal{E}' \to S^\vee - \lambda(C)$ is proper, $\mu^\vee$ is proper and surjective by Proposition 2.16.4.

We will see later that $\sigma^\vee : L \to S^\vee$ is the dual of $\sigma : M \to S$, and $\mu^\vee : \mathcal{E}' \to L$ is the dual of $\mu : \mathcal{E} \to M$.

For every point $p \in L$ by $\hat{T}_p(L)$ we denote the linear subspace of V spanned by the embedded tangent space $\sigma^\vee_* T_p(L)$ and the vector $\sigma^\vee(p)$.

Proposition 3.7. *We consider the inverse image $\mu^{\vee-1}(p)$ of an arbitrary point $p \in L$. We write $\bar{R}_p = \sigma\mu(\mu^{\vee-1}(p))$*

1. *In the case $\epsilon = -1$ of hyperbolic N-space, we have* $\operatorname{rank}(\lambda) = N - 1$ *and* $\bar{R}_p = S \cap \hat{T}_p(L)^\perp$.
2. *In the case $\epsilon = +1$ of an N-sphere, we have two cases.*
   If $\bar{R}_p \neq \tau(\bar{R}_p)$, then $\operatorname{rank}(\lambda) = N - 1$, *$\bar{R}_p$ is a set consisting of a single point, and $\bar{R}_p \cup \tau(\bar{R}_p) = S \cap \hat{T}_p(L)^\perp$.*
   If $\bar{R}_p = \tau(\bar{R}_p)$, then $\bar{R}_p = S \cap \hat{T}_p(L)^\perp$.

Proof. Since $\mathcal{E}$ is compact, $\mu^{\vee-1}(p)$ is compact. Since $\mu^{\vee-1}(p) \subset \mu^{\vee-1}(L) = \mathcal{E}'$, we have $\mu^{\vee-1}(p) \cap C = \emptyset$. Thus the inverse image $\mu^{\vee-1}(p)$ is a smooth compact analytic submanifold in $\mathcal{E}$. Every connected component of $\mu^{\vee-1}(p)$ has dimension equal to $N - 1 - \operatorname{rank}(\lambda)$.

Since $\mu^{\vee-1}(p) \subset \lambda^{-1}(\sigma^\vee(p)) = (\sigma^\vee\mu^\vee)^{-1}(\sigma^\vee(p))$, the subset $\mu(\mu^{\vee-1}(p)) \subset M$ is isomorphic to $\mu^{\vee-1}(p)$ by Lemma 3.2.1. Since $\sigma : M \to S$ is a proper immersion, one can conclude that $\bar{R}_p = \sigma\mu(\mu^{\vee-1}(p)) \subset S$ is a compact analytic set such that every irreducible component has dimension equal to $N - 1 - \operatorname{rank}(\lambda)$.

We show $\bar{R}_p \subset \hat{T}_p(L)^\perp$. Let $\tilde{p} \in \mu^{\vee-1}(p)$ be an arbitrary point. Since $\pi : \mathcal{F} \to \mathcal{E}$ is surjective, we have a point $q \in \mathcal{F}$ with $\pi(q) = \tilde{p}$. We have $(\sigma\mu(\tilde{p}), \sigma^\vee(p)) = (\sigma\mu(\tilde{p}), \sigma^\vee\mu^\vee(\tilde{p})) = (\sigma\mu(\tilde{p}), \lambda(\tilde{p})) = (\sigma\mu\pi(q), \lambda\pi(q)) = (f_0(q), f_N(q)) = 0$. Let $X \in T_p(L)$ be an arbitrary tangent vector. Since $\mu^\vee_* : T_{\tilde{p}}(\mathcal{E}) \to T_p(L)$ and $\pi_* : T_q(\mathcal{F}) \to T_{\tilde{p}}(\mathcal{E})$ is surjective, we have a vector $X' \in T_q(\mathcal{F})$ with $(\mu^\vee\pi)_* X' = X$. $(f_N)_* X' = (\sigma^\vee\mu^\vee\pi)_* X' = \sigma^\vee_* X$. We have $(\sigma\mu(\tilde{p}), \sigma^\vee_* X) = (f_0(q), (f_N)_* X') = \langle (\omega_{0N})_q, X' \rangle$ by Lemma 3.5.3. By Lemma 3.5.2 $\omega_{0N} = -\omega_{N0}$. By Lemma 3.5.4 $\omega_{N0} = 0$. Thus $\omega_{0N} = 0$ and we have $(\sigma\mu(\tilde{p}), \sigma^\vee_* X) = \langle (\omega_{0N})_q, X' \rangle = 0$. Since $\hat{T}_p(L)$ is spanned by the vector $\sigma^\vee(p)$ and $\sigma^\vee_* T_p(L)$, we have $\tilde{p} \in \hat{T}_p(L)^\perp$.

Since $\bar{R}_p \subset S$ we have

$$\bar{R}_p \subset S \cap \hat{T}_p(L)^\perp. \tag{3.1}$$

In particular, we have $S \cap \hat{T}_p(L)^\perp \neq \emptyset$.

Now, Since $\dim T_p(L) = \dim L = \operatorname{rank}(\lambda)$ by Corollary 2.32, we have $\dim \hat{T}_p(L) = \dim T_p(L) + 1 = \operatorname{rank}(\lambda) + 1$. Thus $\dim \hat{T}_p(L)^\perp = (N+1) - (\operatorname{rank}(\lambda) + 1) = N - \operatorname{rank}(\lambda)$, and $\dim S \cap \hat{T}_p(L)^\perp = N - \operatorname{rank}(\lambda) - 1 = \dim \bar{R}_p$.

Consider the case $\epsilon = -1$. In this case $S \cap \hat{T}_p(L)^\perp$ is smooth and connected. Thus the both sides of (3.1) coincide, since they have the same dimension. In particular, $S \cap \hat{T}_p(L)^\perp$ is compact. It implies that $\dim \hat{T}_p(L) = N$ and $\operatorname{rank}(\lambda) = N - 1$ in this case.

Consider the case $\epsilon = +1$ next. In this case $S \cap \hat{T}_p(L)^\perp$ is a sphere with dimension $N - 1 - \operatorname{rank}(\lambda)$, and $S \cap \hat{T}_p(L)^\perp = \tau(S \cap \hat{T}_p(L)^\perp)$. Assume $N - 1 > \operatorname{rank}(\lambda)$. Then, $S \cap \hat{T}_p(L)^\perp$ is smooth and connected. Thus the both sides of (3.1) coincide by the same reason as above. In particular, we have $\bar{R}_p = \tau(\bar{R}_p)$. We consider the last remaining case $N-1 = \mathrm{rank}\,(\lambda)$. In this case $S \cap \hat{T}_p(L)^{\perp}$ is a set of two points. If the both sides of (3.1) coincide, we have $\bar{R}_p = \tau(\bar{R}_p)$. Otherwise $\bar{R}_p$ is a set of a single point, and $\bar{R}_p \cup \tau(\bar{R}_p) = S \cap \hat{T}_p(L)^{\perp}$. □

Proposition 3.8. *In the case $\epsilon = -1$ of hyperbolic N-space, the immersion $\mu^{\vee} : L \to S^{\vee}$ has a special property. For every point $p \in L$ the restriction to $\hat{T}_p(L)$ of the inner product $(\quad,\quad)$ is positive definite.*

Proof. It follows from $S \cap \hat{T}_p(L)^{\perp} \neq \emptyset$. □

Proof of Theorem 3.4. Let L_0 be the set of smooth points of $\bar{L} = \lambda(\mathcal{E}) - \lambda(C)$ and let $X = \{a \in S \mid \text{The vector } a \text{ is orthogonal to } \hat{T}_b(L_0) \text{ for some } b \in L_0.\}$. Note that the induced map $L_0' = \sigma^{\vee -1}(L_0) \to L_0$ by $\sigma^{\vee}$ is an isomorphism and $\mu^{\vee -1}(L_0') = \lambda^{-1}(L_0)$. By $\overline{Z}$ we denote the closure of the set Z.

Consider the case $\epsilon = -1$. Proposition 3.7 implies $X = \sigma\mu(\mu^{\vee -1}(L_0'))$. By definition $(\bar{M}^{\vee})^{\vee} = \overline{X}$. Thus $(\bar{M}^{\vee})^{\vee} = \overline{\sigma\mu(\lambda^{-1}(L_0))}$. Since $\sigma\mu$ is proper, $\overline{\sigma\mu(\lambda^{-1}(L_0))} = \sigma\mu(\overline{\lambda^{-1}(L_0)})$. By Corollary 2.32 $\overline{\lambda^{-1}(L_0)} = \mathcal{E}$. Thus $(\bar{M}^{\vee})^{\vee} = \sigma\mu(\mathcal{E}) = \bar{M}$.

Consider the case $\epsilon = +1$. In this case Proposition 3.7 implies $X = \sigma\mu(\mu^{\vee -1}(L_0')) \cup \tau\sigma\mu(\mu^{\vee -1}(L_0'))$. By the same reasoning as above we have $(\bar{M}^{\vee})^{\vee} = \bar{M} \cup \tau(\bar{M})$. □

Proof of Theorem 1.2. Consider the case $\epsilon = -1$. By Theorem 3.4 we have $\sigma(M) = \sigma'(M')$. Since by Lemma 3.1 $\sigma : M \to S$ and $\sigma' : M' \to S$ can be identified with the normalization of $\sigma(M) = \sigma'(M')$, we have an isomorphism $\phi : M \to M'$ with $\sigma'\phi = \sigma$ by Proposition 2.16.2.

Consider the case $\epsilon = +1$. By Theorem 3.4 we have $\sigma(M) \cup \tau\sigma(M) = \sigma'(M') \cup \tau\sigma'(M')$. Now, by Lemma 2.33.2 $\sigma(M)$ and $\sigma'(M')$ are irreducible. Since $\tau : S \to S$ is an isomorphism $\tau\sigma(M)$ and $\tau\sigma'(M')$ are also irreducible. If $\sigma(M) = \tau\sigma(M)$, then by Theorem 2.14 we have $\sigma'(M') = \tau\sigma'(M')$. Thus by the same reasoning as above one can obtain the conclusion. Assume $\sigma(M) \neq \tau\sigma(M)$. By Theorem 2.14, $\sigma'(M') \neq \tau\sigma'(M')$, and either $\sigma(M) = \sigma'(M')$ or $\sigma(M) = \tau\sigma'(M')$ holds. If $\sigma(M) = \sigma'(M')$, then by the same reasoning one obtains the conclusion. If $\sigma(M) = \tau\sigma'(M')$, applying Proposition 2.16.2 to σ and $\tau\sigma'$, one can conclude the existence of an isomorphism $\phi : M \to M'$ with $\tau\sigma'\phi = \sigma$. □

Proof of Theorem 1.1. Let $g : M \to G(n+1, V)$ and $g' : M' \to G(n+1, V)$ denote the associated Gauss maps. Assume $g(M) = g'(M')$ as sets. Let W_{ξ} denote the $(n+1)$-dimensional linear subspace in V corresponding to a point $\xi \in G(n+1, V)$. By the definition of the dual variety we have $M^{\vee} = S^{\vee} \cap \left(\bigcup_{\xi \in g(M)} W_{\xi}^{\perp}\right)$ and $M'^{\vee} = S^{\vee} \cap \left(\bigcup_{\xi \in g'(M')} W_{\xi}^{\perp}\right)$. By assumption we have $M^{\vee} = M'^{\vee}$. Thus by Theorem 1.2 we have Theorem 1.1. □

Now, we have defined the manifold $\mathcal{E}$ associated with the immersion $\sigma : M \to S$. We can consider the dual object $\mathcal{E}^{\vee}$ of $\mathcal{E}$ associated with the immersion $\sigma^{\vee} : L \to S^{\vee}$. Let

$$\mathcal{E}^{\vee} = \{(p,\, a) \in L \times S \mid a \in S \cap \hat{T}_p(L)^{\perp}\}$$

A map $j : \mathcal{E}' \to L \times S$ can be defined by

$$j(\tilde{p}) = (\mu^\vee(\tilde{p}),\, \sigma\mu(\tilde{p}))$$

for $\tilde{p} \in \mathcal{E}'$. By Proposition 3.7 the image $j(\mathcal{E}')$ is contained in $\mathcal{E}^\vee$. Thus we have a map $j : \mathcal{E}' \to \mathcal{E}^\vee$.

Proposition 3.9. *The map $j : \mathcal{E}' \to \mathcal{E}^\vee$ is injective.*

Proof. Assume first either $\epsilon = -1$, or $\epsilon = +1$ and $\sigma(M) = \tau\sigma(M)$. Let $\mathcal{E}_1^\vee, \mathcal{E}_2^\vee, \ldots$ be connected components of $\mathcal{E}^\vee$. Let $\mathcal{E}_\natural^\vee$ be the union of components $\mathcal{E}_i^\vee$ with $j(\mathcal{E}') \cap \mathcal{E}_i^\vee \neq \emptyset$. We have the induced map $j : \mathcal{E}' \to \mathcal{E}_\natural^\vee$. Let $r_2 : \mathcal{E}_\natural^\vee \to S$ be the map induced by the second projection $L \times S \to S$. By Theorem 3.4 the image $r_2(\mathcal{E}^\vee)$ is contained in $\bar{M} = \sigma(M)$ under our assumption.

We claim here that r_2 is $\bar{M}$-admissible. Let $\mathcal{E}_i^\vee$ be a connected component of $\mathcal{E}_\natural^\vee$. By definition of $\mathcal{E}_\natural^\vee$, $j^{-1}(\mathcal{E}_i^\vee)$ is a non-empty open subset of $\mathcal{E}'$. On the other hand, $\sigma^{-1}(\operatorname{Sing}\bar{M})$ is a closed analytic proper subset of a smooth connected manifold M. Thus $M - \sigma^{-1}(\operatorname{Sing}\bar{M})$ is dense in M. Moreover, since μ is the projection of a fiber bundle, one knows $\mathcal{E}' - (\sigma\mu)^{-1}(\operatorname{Sing}\bar{M})$ is dense in $\mathcal{E}'$. Thus $j^{-1}(\mathcal{E}_i^\vee) - (\sigma\mu)^{-1}(\operatorname{Sing}\bar{M}) \neq \emptyset$. Since $r_2 j = \sigma\mu$ by definition of j, we have $\mathcal{E}_i^\vee - r_2^{-1}(\operatorname{Sing}\bar{M}) \neq \emptyset$. This implies $r_2(\mathcal{E}_i^\vee) \not\subset \operatorname{Sing}\bar{M}$.

Since $\sigma : M \to S$ coincides with the normalization of $\sigma(M)$ we have a map $\mu_2 : \mathcal{E}_\natural^\vee \to M$ with $r_2 = \sigma\mu_2$ by Proposition 2.16. Since $\sigma\mu_2 j = r_2 j = \sigma\mu$, we have $\mu_2 j = \mu$ by the uniqueness in Proposition 2.16.

Let $r_1 : \mathcal{E}^\vee \to L$ denote the map induced by the first projection $L \times S \to L$. By definition of j we have $r_1 j = \mu^\vee$ and $\sigma^\vee r_1 j = \sigma^\vee \mu^\vee = \lambda$.

Let $i : \mathcal{E}_\natural^\vee \to M \times S^\vee$ be a map defined by $i(\tilde{p}) = (\mu_2(\tilde{p}),\, \sigma^\vee r_1(\tilde{p}))$ for $\tilde{p} \in \mathcal{E}_\natural^\vee$. We have $ij(\tilde{p}) = (\mu(\tilde{p}),\, \lambda(\tilde{p})) = \tilde{p}$ for $\tilde{p} \in \mathcal{E}'$. Thus ij is the identity map of $\mathcal{E}'$, and j is injective.

Second, we consider the case where $\epsilon = +1$ and $\sigma(M) \neq \tau\sigma(M)$.

Let M_1 and M_2 be copies of M, and $\tilde{M}$ be the disjoint union of M_1 and M_2. Let $\tilde{\sigma} : \tilde{M} \to S$ be the immersion such that it coincides with σ on M_1 and it coincides with $\tau\sigma$ on M_2.

We here replace M and σ by $\tilde{M}$ and $\tilde{\sigma}$. Then, we obtain new $\mathcal{E}$, L, and $\mathcal{E}^\vee$ and new maps among them. We can apply the same argument as above to the new situation, since after the replacement $\sigma(M) = \tau\sigma(M)$ holds. Thus in the new situation $j : \mathcal{E}' \to \mathcal{E}^\vee$ is injective. Now, it is easy to see that new L and new $\mathcal{E}^\vee$ are same as old ones, old $\mathcal{E}$ is one of two connected components of new $\mathcal{E}$, and new j is the extension of old j. Thus the old j is also injective. □

Corollary 3.10. *For every point $p \in L$ the restriction of $\sigma\mu$ to $\mu^{\vee-1}(p)$ is injective.*

Proof. Since $r_1 j = \mu^\vee$, for every point $p \in L$, an injective map j induces an injective map $\mu^{\vee-1}(p) \to r_1^{-1}(p) \cong S \cap \hat{T}_p(L)^\perp$. The induced map coincides with $\sigma\mu$ by definition of j. □

Theorem 3.11. *We consider the map $\mu^\vee : \mathcal{E}' = \mathcal{E} - \lambda^{-1}(\lambda(C)) \to L$.*

1. *Assume either $\epsilon = -1$, or $\epsilon = +1$ and $\sigma(M) = \tau\sigma(M)$. Then, $\mu^\vee : \mathcal{E}' \to L$ defines a fiber bundle. For every point $p \in L$ the map $\sigma\mu$ induces an isomorphism between the fiber $\mu^{\vee-1}(p)$ and $S \cap \hat{T}_p(L)^\perp$. In particular, if $\epsilon = -1$, then $\dim L = N - 1$ and any fiber $\mu^{\vee-1}(p)$ is a set with only one element.*

2. *Assume* $\epsilon = +1$ *and* $\sigma(M) \neq \tau\sigma(M)$. *Then,* $\dim L = N-1$, *and* $\mu^\vee : \mathcal{E}' \to L$ *is an isomorphism. For every point* $p \in L$, *the fiber* $\mu^{\vee -1}(p)$ *is a set with only one element, and* $\sigma\mu$ *induces a map from* $\mu^{\vee -1}(p)$ *to the set* $S \cap \hat{T}_p(L)^\perp$ *consisting of two points.*

Proof. We write $\bar{R}_p = \sigma\mu(\mu^{\vee -1}(p))$ as above.
1. The case $\epsilon = -1$ follows from Proposition 3.7 and Corollary 3.10.

Assume $\epsilon = +1$ and $\sigma(M) = \tau\sigma(M)$. By Proposition 2.16.1 we have an isomorphism $\hat{\tau} : M \to M$ with $\sigma\hat{\tau} = \tau\sigma$. It is easy to see $\hat{T}_p(M) = \hat{T}_{\hat{\tau}(p)}(M)$ for every $p \in M$. Thus $\hat{\tau} \times id : M \times S^\vee \to M \times S^\vee$ preserves the subset $\mathcal{E}$ and we have an isomorphism $\tilde{\tau} : \mathcal{E} \to \mathcal{E}$ with $\mu\tilde{\tau} = \hat{\tau}\mu$ and $\lambda\tilde{\tau} = \lambda$. By uniqueness in Proposition 2.16.1 we have also $\mu^\vee\tilde{\tau} = \mu^\vee$. Thus for every point $p \in L$, $\mu^{\vee -1}(p) = \tilde{\tau}(\mu^{\vee -1}(p))$. Since $\tau\sigma\mu = \sigma\mu\tilde{\tau}$, we have $\bar{R}_p = \tau(\bar{R}_p)$. By Proposition 3.7.2 $\bar{R}_p = S \cap \hat{T}_p(L)^\perp$ for every $p \in L$. This implies the map $\mu^{\vee -1}(p) \to S \cap \hat{T}_p(L)^\perp$ induced by $\sigma\mu$ is surjective. By Corollary 3.10 it is also injective.
2. By Proposition 3.7.2 we have $\dim L = \operatorname{rank}(\mu^\vee) = \operatorname{rank}(\lambda) = N - 1 = \dim \mathcal{E}'$. Thus $\mu^\vee$ is a proper surjective immersion, and $S \cap \hat{T}_p(L)^\perp$ is a set consisting of two points for every $p \in L$. By Corollary 3.10 $\sigma\mu : \mu^{\vee -1}(p) \to S \cap \hat{T}_p(L)^\perp$ is injective, and $\mu^{\vee -1}(p)$ contains at most two points for every $p \in L$.

Let U denote the set of points $p \in L$ such that $\mu^{\vee -1}(p)$ contains only one point.

Assume $p \notin U$. Then, $\bar{R}_p = \sigma\mu(\mu^{\vee -1}(p))$ contains two points, and $\bar{R}_p = S \cap \hat{T}_p(L)^\perp$. Thus $\bar{R}_p = \tau(\bar{R}_p)$. Let p_1 be one of two points in $\bar{R}_p$. We have $\tau(p_1) \in \bar{R}_p$. Since $\bar{R}_p \subset \sigma(M)$, we have $p_1 \in \Sigma = \sigma(M) \cap \tau\sigma(M)$, and $p \in \bar{\Sigma} = \mu^\vee((\sigma\mu)^{-1}(\Sigma) \cap \mathcal{E}')$. One knows $L - \bar{\Sigma} \subset U$.

Since $(\sigma\mu)^{-1}(\Sigma)$ is a closed analytic set such that $\mathcal{E} - (\sigma\mu)^{-1}(\Sigma)$ is dense in $\mathcal{E}$, and since $\mu^\vee$ is a proper immersion, $\bar{\Sigma}$ is a closed analytic set in L such that $L - \bar{\Sigma}$ is dense in L. Thus U is not empty and is dense in L. On the other hand, the number of points in $\mu^{\vee -1}(p)$ is a locally constant as a function of $p \in L$, since $\mu^\vee$ is a proper immersion. We conclude $L = U$, and $\mu^\vee$ is injective. □

By Theorem 3.11 we can say that the diagram

$$V \supset S^\vee \xleftarrow{\sigma^\vee} L \xleftarrow{\mu^\vee} \mathcal{E}' \xrightarrow{\mu} M \xrightarrow{\sigma} S \subset V \tag{3.2}$$

is symmetric with respect to the center $\mathcal{E}'$.

We collect the definitions here.

$S^\vee = \{a \in V \mid (a, a) = 1\}$
$\hat{T}_p(M) = \mathbf{R}\sigma(p) + \sigma_* T_p(M)$: a linear subspace of V associated with a point $p \in M$
$Z^\perp$: the orthogonal complement of a subset Z in V
$\mathcal{E} = \{(p, a) \in M \times S^\vee \mid a \in S^\vee \cap \hat{T}_p(M)^\perp\}$
$\mu : \mathcal{E} \to M$: the first projection
$\lambda : \mathcal{E} \to S^\vee$: the second projection
C: the critical set of λ
$\mathcal{E}' = \mathcal{E} - \lambda^{-1}(\lambda(C))$
$\sigma^\vee : L \to S^\vee$: the normalization of $\bar{L} = \lambda(\mathcal{E}) - \lambda(C)$
$\mu^\vee : \mathcal{E}' \to L$: the induced map. $\sigma^\vee\mu^\vee = \lambda \mid \mathcal{E}'$

Corollary 3.12. 1. *If a pair* (p, a) *of a point* $p \in M$ *and a normal vector* $a \in S^\vee$ *of* M *at* p *in* S *belongs to the dense subset* $\mathcal{E}' = \mathcal{E} - \lambda^{-1}(\lambda(C))$ *of the set* $\mathcal{E}$ *of* *all such pairs, then there is a point $q \in L$ such that $a = \sigma^\vee(q)$ and $\sigma(p) \in S$ is a normal vector of L at q in $S^\vee$.*

2. *For every pair (q, b) of a point $q \in L$ and a normal vector $b \in S$ of L at q in $S^\vee$, there is a point $p \in M$ such that $\pm b = \sigma(p)$ and $\sigma^\vee(q)$ is a normal vector of M at p in S.*

4. Metric study of the dual variety

In this section we study properties of the dual variety related to the Riemannian metric. We use the same notations as in the previous section. Recall the following commutative diagram:

$$\begin{array}{ccccccc} \mathcal{F} & = & \mathcal{F} & = & \mathcal{F} & = & \mathcal{F} \\ \downarrow f_N & & \downarrow \pi & & \downarrow \rho & & \downarrow f_0 \\ V \supset S^\vee & \xleftarrow{\lambda} & \mathcal{E} & \xrightarrow{\mu} & M & \xrightarrow{\sigma} & S \subset V \end{array}$$

For simplicity we use the ranges of indices

$$1 \le i, j, k \le n; \quad n+1 \le \zeta, \eta, \theta \le N.$$

First, we introduce the second fundamental form.

Lemma 4.1. *The Maurer-Cartan form ω on $\mathcal{F}$ has the following properties:*

1. *$R_g^* \omega = {}^t g \omega g$ for every $g \in O(n) \times O(N-n)$, where $g = (g', g'')$ ($g' \in O(n)$, $g'' \in O(N-n)$) is identified with an element in $\mathrm{Hom}(\mathbf{R}^{N+1}, \mathbf{R}^{N+1})$ represented by the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & g' & 0 \\ 0 & 0 & g'' \end{pmatrix}$.*
2. *Let $d\tilde{s}^2$ denote the metric on S induced by the inner product $(\ ,\ )$. We have $f_0^* d\tilde{s}^2 = \sum_i (\omega_{i0})^2$.*
3. *One-forms $(\omega_{10})_q, (\omega_{20})_q, \ldots, (\omega_{n0})_q \in T_q^*(\mathcal{F})$ are linearly independent at every point $q \in \mathcal{F}$*

Lemma 4.2. *There are functions $A_{\zeta ij}$ on $\mathcal{F}$ satisfying $\omega_{\zeta i} = \sum_j A_{\zeta ij} \omega_{j0}$ and $A_{\zeta ij} = A_{\zeta ji}$.*

Proof. By Lemma 3.5.4 $\omega_{\zeta 0} = 0$. By Lemma 3.5.2 $\omega_{00} = 0$. Thus by Lemma 3.5.1 we have $0 = -d\,\omega_{\zeta 0} = \epsilon\,\omega_{\zeta 0} \wedge \omega_{00} + \sum_i \omega_{\zeta i} \wedge \omega_{i0} + \sum_\eta \omega_{\zeta\eta} \wedge \omega_{\eta 0} = \sum_i \omega_{\zeta i} \wedge \omega_{i0}$. Our lemma follows from Lemma 4.1.3. □

Corollary 4.3. *Let Λ_ζ be the section of the vector bundle $T^*(\mathcal{F}) \otimes T^*(\mathcal{F})$ over $\mathcal{F}$ defined by $\Lambda_\zeta = \sum_j \omega_{\zeta j} \otimes \omega_{j0} = \sum_i \sum_j A_{\zeta ij} \omega_{i0} \otimes \omega_{j0}$.*

1. *Λ_ζ is symmetric, in other words, $\langle (\Lambda_\zeta)_q, X \otimes Y \rangle = \langle (\Lambda_\zeta)_q, Y \otimes X \rangle$ for every $q \in \mathcal{F}$ and for every $X, Y \in T_q(\mathcal{F})$.*
2. *Λ_ζ is horizontal, in other words, $\langle (\Lambda_\zeta)_q, X \otimes Y \rangle = 0$ if $\rho_* X = 0$ or $\rho_* Y = 0$ for every $q \in \mathcal{F}$ and for every $X, Y \in T_q(\mathcal{F})$.*
3. *$\langle (\Lambda_\zeta)_q, X \otimes Y \rangle = -((f_\zeta)_* X, (f_0)_* Y)$ for every $q \in \mathcal{F}$ and for every $X, Y \in T_q(\mathcal{F})$.*

Corollary 4.4. *Let Λ be the section of the vector bundle $V \otimes T^*(\mathcal{F}) \otimes T^*(\mathcal{F})$ over $\mathcal{F}$ defined by $\Lambda = \sum_\zeta f_\zeta \Lambda_\zeta$.*

1. *Λ is symmetric.*
2. *Λ is horizontal.*

3. $\langle \Lambda_q,\, X \otimes Y\rangle = -\sum_\zeta f_\zeta(q)((f_\zeta)_* X,\, (f_0)_* Y)$ *for every* $q \in \mathcal{F}$ *and for every* $X, Y \in T_q(\mathcal{F})$.
4. Λ *is* $O(n) \times O(N-n)$*-invariant, in other words,* $R_g^* \Lambda = \Lambda$ *for every* $g \in O(n) \times O(N-n)$.

Let $S^2T(M)$ denote the symmetric product of degree 2 of the tangent bundle $T(M)$ of M, and $N(M/S)$ denote the normal bundle of M in S. They are vector bundles over M. The fiber $N_p(M/S)$ over $p \in M$ of the bundle $N(M/S)$ can be identified with the orthogonal complement of $\hat{T}_p(M)$ in V.

Corollary 4.5. *There is a map* $II : S^2T(M) \to N(M/S)$ *of vector bundles satisfying* $II(\rho_* X \cdot \rho_* Y) = \langle \Lambda_q,\, X \otimes Y\rangle$ *for every* $q \in \mathcal{F}$ *and for every* $X, Y \in T_q(\mathcal{F})$.

The map $II : S^2T(M) \to N(M/S)$ is the *second fundamental form* of M.

A map $\tilde{II} : S^2(\mu^* T(M)) \to \mathbf{R}$ of bundles over $\mathcal{E}$ is defined by $\tilde{II}(X \cdot Y) = (II(X \cdot Y),\, \lambda(\tilde{p}))$ for a point $\tilde{p} \in \mathcal{E}$ and $X, Y \in (\mu^* T(M))_{\tilde{p}}$. Under the canonical identification $(\mu^* T(M))_{\tilde{p}} = T_{\mu(\tilde{p})}(M)$ the induced map $S^2 T_{\mu(\tilde{p})}(M) \to \mathbf{R}$ is the *second fundamental form at* $p = \mu(\tilde{p}) \in M$ *in the normal direction* $\lambda(\tilde{p})$. By definition $\tilde{II}(\rho_* X,\, \rho_* Y) = \langle (\Lambda_N)_q,\, X \otimes Y\rangle = -((f_N)_* X,\, (f_0)_* Y)$ for every $q \in \mathcal{F}$ and $X, Y \in T_q(\mathcal{F})$.

For every point $\tilde{p} \in \mathcal{E}$ a subspace

$$\operatorname{rad}_{\tilde{p}}(\tilde{II}) = \{X \in T_{\mu(\tilde{p})}(M) \mid \tilde{II}(X,\, Y) = 0 \text{ for every } Y \in T_{\mu(\tilde{p})}(M)\}$$

is called the *radical* at $\tilde{p}$ of the bilinear form $\tilde{II}$. The rank of the symmetric bilinear form $T_{\mu(\tilde{p})}(M) \times T_{\mu(\tilde{p})}(M) \to \mathbf{R}$ induced by $\tilde{II}$ is denoted by $\operatorname{rank}_{\tilde{p}}(\tilde{II})$. We have $\dim \operatorname{rad}_{\tilde{p}}(\tilde{II}) + \operatorname{rank}_{\tilde{p}}(\tilde{II}) = \dim T_{\mu(\tilde{p})}(M) = n$.

Lemma 4.6. $\operatorname{rank}(\lambda_* : T_{\tilde{p}}(\mathcal{E}) \to V) = N - n - 1 + \operatorname{rank}_{\tilde{p}}(\tilde{II})$ *for every point* $\tilde{p} \in \mathcal{E}$.

Proof. Let $q \in \mathcal{F}$ be an arbitrary point. We write $\bar{P}_q = \lambda(\mu^{-1}(\rho(q)))$. By definition $\bar{P}_q = S^\vee \cap \hat{T}_{\rho(q)}(M)^\perp$. Thus we have $V = \mathbf{R} f_N(q) + T_{f_N(q)}(\bar{P}_q) + \hat{T}_{\rho(q)}(M)$ (orthogonal direct sum), and $T_{f_N(q)}(S^\vee) = T_{f_N(q)}(\bar{P}_q) + \hat{T}_{\rho(q)}(M)$. On the other hand, we have an exact sequence $0 \to T_{\pi(q)}(\mu^{-1}(\rho(q))) \to T_{\pi(q)}(\mathcal{E}) \to T_{\rho(q)}(M) \to 0$. Note that $\operatorname{rank}(\lambda_* : T_{\tilde{p}}(\mathcal{E}) \to V) = \operatorname{rank}(\lambda_* : T_{\tilde{p}}(\mathcal{E}) \to T_{f_N(q)}(S^\vee))$. We consider the restriction of $\lambda_* : T_{\pi(q)}(\mathcal{E}) \to T_{f_N(q)}(S^\vee)$ to the subspace $T_{\pi(q)}(\mu^{-1}(\rho(q)))$. By Lemma 3.2.2 the restriction of λ_* to $T_{\pi(q)}(\mu^{-1}(\rho(q)))$ is the isomorphism into $T_{f_N(q)}(\bar{P}_q)$. Let

$$\bar{\lambda}_* : T_{\rho(q)}(M) = T_{\pi(q)}(\mathcal{E})/T_{\pi(q)}(\mu^{-1}(\rho(q))) \to \hat{T}_{\rho(q)}(M) = T_{f_N(q)}(S^\vee)/\hat{T}_{f_N(q)}(\bar{P}_q)$$

denote the induced map. We have $\operatorname{rank}(\lambda_*) = \operatorname{rank}(\bar{\lambda}_*) + N - n - 1$, since $\dim T_{\pi(q)}(\mu^{-1}(\rho(q))) = N - n - 1$.

Let $X_1, X_2, \ldots, X_n \in T_q(\mathcal{F})$ be tangent vectors with $\langle (\omega_{i0})_q,\, X_i\rangle = 1$ and $\langle (\omega_{i0})_q,\, X_j\rangle = 0$ for $i \neq j$. By Lemma 4.1.3 such tangent vectors exist. By Lemma 3.5.3 and 4 one knows $(f_0)_* X_i = f_i(q)$. In particular, $\rho_* X_1, \rho_* X_2, \ldots, \rho_* X_n$ are a basis of $T_{\rho(q)}(M)$, since $(f_0)_* X_i = \sigma_* \rho_* X_i$. On the other hand, $f_0(q), f_1(q), f_2(q), \ldots, f_n(q)$ are a basis of $\hat{T}_{\rho(q)}(M)$. We compute the matrices of $\bar{\lambda}_*$ and $\tilde{II}$ with respect to these bases.

By definition $\bar{\lambda}_* \rho_* X_i = \lambda_* \pi_* X_i = (f_N)_* X_i$. By Lemma 3.5.3 $(f_0(q),\, (f_N)_* X_i) = \langle (\omega_{0N})_q,\, X_i\rangle$ and $(f_j(q),\, (f_N)_* X_i) = \langle (\omega_{jN})_q,\, X_i\rangle$. By Lemma 3.5.2 and 4 $\omega_{0N} = 0$.

Thus $(f_0(q),\, \bar{\lambda}_*\rho_* X_i) = 0$. By Lemma 3.5.2 and Lemma 4.2 $\omega_{jN} = -\sum_k A_{Njk}\omega_{k0}$. Thus $\langle(\omega_{jN})_q,\, X_i\rangle = -\sum_k A_{Njk}(q)\langle(\omega_{k0})_q,\, X_i\rangle = -A_{Nji}(q)$. We can conclude $(f_j(q),\, \bar{\lambda}_*\rho_* X_i) = -A_{Nji}(q)$.

We compute the matrix of $\tilde{II}$. One has $\tilde{II}(\rho_* X_i,\, \rho_* X_j) = -((f_N)_* X_i,\, (f_0)_* X_j)$ $= -((f_N)_* X_i,\, f_j(q)) = -\langle(\omega_{jN})_q,\, X_i\rangle = \sum_k A_{Njk}(q)\langle(\omega_{k0})_q,\, X_i\rangle = A_{Nji}(q)$. One knows that the two matrices are essentially same. Since $\pi : \mathcal{F} \to \mathcal{E}$ is surjective, we obtain our lemma. □

Let $\operatorname{rank}(\tilde{II}) = \max_{\tilde{p}\in\mathcal{E}} \operatorname{rank}_{\tilde{p}}(\tilde{II})$.

Corollary 4.7. 1. $\operatorname{rank}(\lambda) = N - n - 1 + \operatorname{rank}(\tilde{II})$.
2. $C = \{\tilde{p} \in \mathcal{E} \mid \operatorname{rank}_{\tilde{p}}(\lambda) < \operatorname{rank}(\lambda)\} = \{\tilde{p} \in \mathcal{E} \mid \operatorname{rank}_{\tilde{p}}(\tilde{II}) < \operatorname{rank}(\tilde{II})\}$.

Let $m = n - \operatorname{rank}(\tilde{II})$. For every $\tilde{p} \in \mathcal{E} - C$ we have $\dim \operatorname{rad}_{\tilde{p}}(\tilde{II}) = m$.

Below we use the ranges of indices

$$0 \le E, F, G \le N;$$
$$1 \le i, j, k \le n; \quad m+1 \le r, s, t \le N-1;$$
$$1 \le I, J, K \le m; \quad m+1 \le \alpha, \beta, \gamma \le n; \quad n+1 \le R, S, T \le N-1.$$

Recall here that we have a symmetric diagram (3.2). However, our frame manifold $\mathcal{F}$ represents only information of the right half of (3.2).

Therefore we introduce another manifold $\mathcal{G}$ and several maps. Put

$$\mathcal{G} = \{q \in \mathcal{F} \mid \pi(q) \in \mathcal{E} - \lambda^{-1}(\lambda(C)),$$
$$f_1(q), f_2(q), \ldots, f_m(q) \text{ are a basis of } \operatorname{rad}_{\pi(q)}(\tilde{II})\}.$$

Let $\iota : \mathcal{G} \to \mathcal{F}$ denote the inclusion map. We denote $\tilde{f}_E = f_E\iota$, $\tilde{\pi} = \pi\iota$, $\tilde{\rho} = \rho\iota$, $\tilde{A}_{Rij} = A_{Rij}\iota$, and $\tilde{A}_{Nij} = A_{Nij}\iota$. The map $\tilde{\pi} : \mathcal{G} \to \mathcal{E}' = \mathcal{E} - \lambda^{-1}(\lambda(C))$ is the projection of a principal $O(m) \times O(n-m) \times O(N-n-1)$-bundle. Thus $\mathcal{G}$ is a smooth analytic manifold, and we can consider the Maurer-Cartan forms $\tilde{\omega} = \iota^*\omega$, $\tilde{\omega}_{EF} = \iota^*\omega_{EF}$ on $\mathcal{G}$.

We obtain the commutative diagram below.

$$\begin{array}{ccccccccc}
\mathcal{G} & = & \mathcal{G} & = & \mathcal{G} & = & \mathcal{G} & = & \mathcal{G} \\
\downarrow{\scriptstyle \tilde{f}_N} & & \downarrow{\scriptstyle \tilde{\rho}^\vee} & & \downarrow{\scriptstyle \tilde{\pi}} & & \downarrow{\scriptstyle \tilde{\rho}} & & \downarrow{\scriptstyle \tilde{f}_0} \\
V \supset S^\vee & \xleftarrow{\sigma^\vee} & L & \xleftarrow{\mu^\vee} & \mathcal{E}' & \xrightarrow{\mu} & M & \xrightarrow{\sigma} & S \subset V
\end{array}$$

Later in Lemma 4.9 we will see that this diagram is symmetric with respect to the central column $\tilde{\pi} : \mathcal{G} \to \mathcal{E}'$.

To make sure we write the definitions here again.

$\mathcal{G}$: the set of points $(p, a_0, a_1, \ldots, a_N) \in M \times V^{N+1}$ satisfying
1. $a_0 = \sigma(p)$
2. $(a_E,\, a_E) = 1$ for $E \ge 1$, $(a_E,\, a_F) = 0$ for $E \ne F$
3. $a_1, a_2, \ldots, a_n$ are a basis of $T_p(M)$.
4. $\tilde{p} = (p, a_N) \in \mathcal{E}'$.
5. $a_1, a_2, \ldots, a_m$ are a basis of $\operatorname{rad}_{\tilde{p}}(\tilde{II})$ for $\tilde{p} = (p, a_N)$.

$\tilde{f}_E : \mathcal{G} \to V$: a map. $\tilde{f}_E(\tilde{q}) = a_E$ for $\tilde{q} = (p, a_0, a_1, \ldots, a_N) \in \mathcal{G}$.

$\tilde{f} : \mathcal{G} \to \operatorname{Hom}(\mathbf{R}^{N+1},\, V)$: For $\tilde{q} \in \mathcal{G}$, $\tilde{f}(\tilde{q}) \in \operatorname{Hom}(\mathbf{R}^{N+1},\, V)$ is the linear map satisfying $\tilde{f}(\tilde{q})(x) = \sum_E x_E \tilde{f}_E(\tilde{q})$ for $x = (x_0, x_1, \ldots, x_N) \in \mathbf{R}^{N+1}$

$\tilde{\omega} = I\tilde{f}^{-1}d\tilde{f}$: the Maurer-Cartan form. a Hom($\mathbf{R}^{N+1}$, $\mathbf{R}^{N+1}$)-valued differential 1-form on $\mathcal{G}$.
$\tilde{\omega}_{EF}$: a matrix entry of $\tilde{\omega}$.
$\tilde{\rho} : \mathcal{G} \to M$: the first projection.
$\tilde{\pi} : \mathcal{G} \to \mathcal{E}'$: $\tilde{\pi}(\tilde{q}) = (p, a_N)$ for $\tilde{q} = (p, a_0, a_1, \ldots, a_N) \in \mathcal{G}$.
$\tilde{\rho}^\vee = \tilde{\pi}\mu^\vee : \mathcal{G} \to L$
$\tilde{A}_{Rij}$, $\tilde{A}_{Nij}$: functions on $\mathcal{G}$. $\tilde{\omega}_{Ri} = \sum_j \tilde{A}_{Rij}\tilde{\omega}_{j0}$. $\tilde{\omega}_{Ni} = \sum_j \tilde{A}_{Nij}\tilde{\omega}_{j0}$.

Lemma 4.8. 1. *$d\tilde{\omega} + \tilde{\omega} \wedge I\tilde{\omega} = 0$.*
2. *${}^t\tilde{\omega} = -\tilde{\omega}$.*
3. *For every E, F for every point $\tilde{q} \in \mathcal{G}$ and for every tangent vector $X \in T_{\tilde{q}}(\mathcal{G})$*

$$\langle (\tilde{\omega}_{EF})_{\tilde{q}},\, X\rangle = (\tilde{f}_E(\tilde{q}),\, (\tilde{f}_F)_* X).$$

4. *$\tilde{I}I(\tilde{\rho}_*X,\, \tilde{\rho}_*Y) = -((\tilde{f}_N)_*X,\, (\tilde{f}_0)_*Y)$ for any $\tilde{q} \in \mathcal{G}$ and $X, Y \in T_{\tilde{q}}(\mathcal{G})$.*
5. *$\tilde{\omega}_{R0} = 0$, $\tilde{\omega}_{N0} = 0$.*
6. *One-forms $(\tilde{\omega}_{10})_{\tilde{q}}, (\tilde{\omega}_{20})_{\tilde{q}}, \ldots, (\tilde{\omega}_{n0})_{\tilde{q}} \in T^*_{\tilde{q}}(\mathcal{G})$ are linearly independent at every point $\tilde{q} \in \mathcal{G}$.*
7. *$\tilde{\omega}_{Ri} = \sum_j \tilde{A}_{Rij}\tilde{\omega}_{j0}$. $\tilde{A}_{Rij} = \tilde{A}_{Rji}$. $\tilde{\omega}_{Ni} = \sum_j \tilde{A}_{Nij}\tilde{\omega}_{j0}$. $\tilde{A}_{Nij} = \tilde{A}_{Nji}$.*
8. *For every point $\tilde{q} \in \mathcal{G}$ we have tangent vectors $X_1, X_2, \ldots, X_n \in T_{\tilde{q}}(\mathcal{G})$ with $\langle (\tilde{\omega}_{i0})_{\tilde{q}},\, X_i\rangle = 1$ and $\langle (\tilde{\omega}_{i0})_{\tilde{q}},\, X_j\rangle = 0$ for $i \neq j$. They satisfy $(\tilde{f}_0)_*X_i = \tilde{f}_i(\tilde{q})$ and $\tilde{I}I(\tilde{\rho}_*X_i,\, \tilde{\rho}_*X_j) = \tilde{A}_{Nij}(\tilde{q})$.*
9. *$\tilde{A}_{NIj} = 0$.*
10. *$\tilde{\omega}_{IN} = 0$, $\tilde{\omega}_{0N} = 0$.*
11. *One-forms $(\tilde{\omega}_{m+1,N})_{\tilde{q}}, (\tilde{\omega}_{m+2,N})_{\tilde{q}}, \ldots, (\tilde{\omega}_{N-1,N})_{\tilde{q}} \in T^*_{\tilde{q}}(\mathcal{G})$ are linearly independent at every point $\tilde{q} \in \mathcal{G}$.*
12. *There are functions $\tilde{B}_{Irs}$, $\tilde{B}_{0rs}$ on $\mathcal{G}$ satisfying $\tilde{\omega}_{Ir} = \sum_s \tilde{B}_{Irs}\tilde{\omega}_{sN}$, $\tilde{B}_{Irs} = \tilde{B}_{Isr}$, $\tilde{\omega}_{0r} = \sum_s \tilde{B}_{0rs}\tilde{\omega}_{sN}$, and $\tilde{B}_{0rs} = \tilde{B}_{0sr}$.*
13. *$\tilde{B}_{0Rs} = 0$.*
14. *The $(n-m) \times (n-m)$-matrix $(\tilde{B}_{0\alpha\beta})$ is the inverse matrix of $(\tilde{A}_{N\alpha\beta})$.*

Proof. 1, 2, 3, 4, 5, 6, 7, 8. See Lemma 3.5, Lemma 4.1, Lemma 4.2 and Corollary 4.3.
9. Let $\tilde{q} \in \mathcal{G}$ be an arbitrary point. Let $X_1, X_2, \ldots, X_n \in T_{\tilde{q}}(\mathcal{G})$ be the vectors in 8.

Now, $\sigma_*\tilde{\rho}_*X_I = \tilde{f}_I(\tilde{q}) \in \sigma_*\mathrm{rad}_{\tilde{\pi}(\tilde{q})}(\tilde{I}I)$. Since $\sigma_* : T_{\tilde{\rho}(\tilde{q})}(M) \to V$ is injective, we have $\tilde{\rho}_*X_I \in \mathrm{rad}_{\tilde{\pi}(\tilde{q})}(\tilde{I}I)$, and $\tilde{A}_{NIj}(\tilde{q}) = \tilde{I}I(\tilde{\rho}_*X_I,\, \tilde{\rho}_*X_j) = 0$. Thus $\tilde{A}_{NIj} = 0$.
10. $\tilde{\omega}_{IN} = -\tilde{\omega}_{NI} = -\sum_j \tilde{A}_{NIj}\tilde{\omega}_{j0} = 0$ by 9. $\tilde{\omega}_{0N} = -\tilde{\omega}_{N0} = 0$ by 5.
11. By 10 $(\tilde{f}_N)_*Y = \sum_\alpha \langle (\tilde{\omega}_{\alpha N})_{\tilde{q}},\, Y\rangle \tilde{f}_\alpha(\tilde{q}) + \sum_R \langle (\tilde{\omega}_{RN})_{\tilde{q}},\, Y\rangle \tilde{f}_R(\tilde{q})$ for $Y \in T_{\tilde{q}}(\mathcal{G})$. The right-hand side is contained in the $(N-m-1)$-dimensional linear space F spanned by $\tilde{f}_{m+1}(\tilde{q}), \tilde{f}_{m+2}(\tilde{q}), \ldots, \tilde{f}_{N-1}(\tilde{q})$.

On the other hand, $\tilde{\pi}_* : T_{\tilde{q}}(\mathcal{G}) \to T_{\tilde{\pi}(\tilde{q})}(\mathcal{E})$ is surjective, and $(\tilde{f}_N)_* = \lambda_*\tilde{\pi}_*$. Thus $\mathrm{rank}\,((\tilde{f}_N)_* : T_{\tilde{q}}(\mathcal{G}) \to V) = \mathrm{rank}\,(\lambda_* : T_{\tilde{\pi}(\tilde{q})}(\mathcal{E}) \to V) = \mathrm{rank}_{\tilde{\pi}(\tilde{q})}(\lambda) = N - m - 1$.

One can conclude that every vector in F is equal to $(\tilde{f}_N)_*Y$ for some $Y \in T_{\tilde{q}}(\mathcal{G})$. It implies claim 11.
12. By 1 and 10 $\sum_r \tilde{\omega}_{Ir} \wedge \tilde{\omega}_{rN} = -d\,\tilde{\omega}_{IN} = 0$. By 11 one knows the former half. The case of $\tilde{B}_{0rs}$ is similar.
13. By 2, 5 and 12 $0 = \tilde{\omega}_{0R} = \sum_s \tilde{B}_{0Rs}\tilde{\omega}_{sN}$. By 11 we obtain claim 13.

14. By 2, 7, 9, 12, 13 we have $\tilde{\omega}_{\alpha 0} = \sum_\beta \sum_\gamma \tilde{B}_{0\alpha\beta}\tilde{A}_{N\beta\gamma}\tilde{\omega}_{\gamma 0}$. By 6 we obtain claim 14. □

The second fundamental form of the immersion $\sigma^\vee : L \to S^\vee$ is denoted by

$$II^\vee : S^2T(L) \to N(L/S^\vee).$$

Let $p \in L$ be a point and $Z \in N_p(L/S^\vee)$ be a normal vector of L at p in $S^\vee$. The bilinear form $T_p(L) \times T_p(L) \to \mathbf{R}$ defined by $\tilde{II}^\vee(X \cdot Y) = (II^\vee(X \cdot Y), Z)$ for $X, Y \in T_p(L)$ is called the *second fundamental form* of L at p *in the normal direction* Z.

On the other hand, we have a map

$$\tilde{II}^\vee : S^2(\mu^{\vee *}T(L)) \to \mathbf{R}$$

of bundles over $\mathcal{E}' = \mathcal{E} - \lambda^{-1}(\lambda(C))$ defined by $\tilde{II}^\vee(X \cdot Y) = (II^\vee(X \cdot Y), \sigma\mu(\tilde{p}))$ for a point $\tilde{p} \in \mathcal{E}'$ and $X, Y \in (\mu^{\vee *}T(L))_{\tilde{p}}$. Under the canonical identification $(\mu^{\vee *}T(L))_{\tilde{p}} = T_{\mu^\vee(\tilde{p})}(L)$ the induced map $S^2T_{\mu^\vee(\tilde{p})}(L) \to \mathbf{R}$ is the second fundamental form at $p = \mu^\vee(\tilde{p}) \in L$ in the normal direction $\sigma\mu(\tilde{p})$.

Note that by Theorem 3.11, for every point $p \in L$ and for every normal vector $Z \in N_p(L/S^\vee)$ of L at p in $S^\vee$, there are a point $\tilde{p} \in \mathcal{E}'$ and a real number $A \in \mathbf{R}$ with $p = \mu^\vee(\tilde{p})$ and $Z = A\sigma\mu(\tilde{p})$. Therefore we lose no essential information in the study of the second fundamental form of L, considering only $\tilde{II}^\vee : S^2(\mu^{\vee *}T(L)) \to \mathbf{R}$.

For every point $\tilde{p} \in \mathcal{E}'$ a subspace

$$\mathrm{rad}_{\tilde{p}}(\tilde{II}^\vee) = \{X \in T_{\mu^\vee(\tilde{p})}(L) \mid \tilde{II}^\vee(X, Y) = 0 \text{ for every } Y \in T_{\mu^\vee(\tilde{p})}(L)\}$$

is the *radical* at $\tilde{p}$ of the bilinear form $\tilde{II}^\vee$. The rank of the symmetric bilinear form $T_{\mu^\vee(\tilde{p})}(L) \times T_{\mu^\vee(\tilde{p})}(L) \to \mathbf{R}$ induced by $\tilde{II}^\vee$ is denoted by $\mathrm{rank}_{\tilde{p}}(\tilde{II}^\vee)$. We have $\dim \mathrm{rad}_{\tilde{p}}(\tilde{II}^\vee) + \mathrm{rank}_{\tilde{p}}(\tilde{II}^\vee) = \dim T_{\mu^\vee(\tilde{p})}(L) = \mathrm{rank}(\lambda) = N - m - 1$. By $\mathrm{rank}(\tilde{II}^\vee)$ we denote the maximum of $\mathrm{rank}_{\tilde{p}}(\tilde{II}^\vee)$ for $\tilde{p} \in \mathcal{E}'$

Lemma 4.9. *Let $\tilde{q} \in \mathcal{G}$ be an arbitrary point.*

1. *Vectors $\tilde{f}_{m+1}(\tilde{q}), \tilde{f}_{m+2}(\tilde{q}), \ldots, \tilde{f}_{N-1}(\tilde{q})$ are a basis of $T_{\tilde{\rho}^\vee(\tilde{q})}(L)$.*
2. *$\tilde{II}^\vee(\tilde{\rho}^\vee_* X, \tilde{\rho}^\vee_* Y) = -((\tilde{f}_0)_* X, (\tilde{f}_N)_* Y)$ for any $X, Y \in T_{\tilde{q}}(\mathcal{G})$.*
3. *We have tangent vectors $Y_{m+1}, Y_{m+2}, \ldots, Y_{N-1} \in T_{\tilde{q}}(\mathcal{G})$ with $\langle(\tilde{\omega}_{rN})_{\tilde{q}}, Y_r\rangle = 1$ and $\langle(\tilde{\omega}_{rN})_{\tilde{q}}, Y_s\rangle = 0$ for $r \neq s$. They satisfy $(\tilde{f}_N)_* Y_r = \tilde{f}_r(\tilde{q})$ and $\tilde{II}^\vee(\tilde{\rho}^\vee_* Y_r, \tilde{\rho}^\vee_* Y_s) = \tilde{B}_{0rs}(\tilde{q})$.*
4. *Vectors $\tilde{f}_{n+1}(\tilde{q}), \tilde{f}_{n+2}(\tilde{q}), \ldots, \tilde{f}_{N-1}(\tilde{q})$ are a basis of $\mathrm{rad}_{\tilde{\pi}(\tilde{q})}(\tilde{II}^\vee)$.*

Proof. 1. Since the critical set of $\tilde{\rho}^\vee = \mu^\vee\pi^\vee$ is the empty set, one knows that $\tilde{\rho}^\vee_* : T_{\tilde{q}}(\mathcal{G}) \to T_{\tilde{\rho}^\vee(\tilde{q})}(L)$ is surjective. Thus $T_{\tilde{\rho}^\vee(\tilde{q})}(L) = \mathrm{Im}((\tilde{f}_N)_* : T_{\tilde{q}}(\mathcal{G}) \to V)$. By Lemma 4.8.10 we have $(\tilde{f}_N)_* X = \sum_r \langle(\tilde{\omega}_{rN})_{\tilde{q}}, X\rangle \tilde{f}_r(\tilde{q})$ for any $X \in T_{\tilde{q}}(\mathcal{G})$. By Lemma 4.8.11 we obtain claim 1.
2. By definition we have $\tilde{II}^\vee(\tilde{\rho}^\vee_* X, \tilde{\rho}^\vee_* Y) = \langle(\sum_r \tilde{\omega}_{0r} \otimes \tilde{\omega}_{rN})_{\tilde{q}}, X \otimes Y\rangle = -\sum_r \langle(\tilde{\omega}_{r0})_{\tilde{q}}, X\rangle\langle(\tilde{\omega}_{rN})_{\tilde{q}}, Y\rangle$. Since $(\tilde{f}_N)_* Y = \sum_r \langle(\tilde{\omega}_{rN})_{\tilde{q}}, Y\rangle \tilde{f}_r(\tilde{q})$, and since $(\tilde{f}_0)_* X = \sum_i \langle(\tilde{\omega}_{i0})_{\tilde{q}}, X\rangle \tilde{f}_i(\tilde{q})$ by Lemma 4.8.5, one knows that the last term is equal to $-((\tilde{f}_0)_* X, (\tilde{f}_N)_* Y)$.
3. The existence of $Y_{m+1}, Y_{m+2}, \ldots, Y_{N-1} \in T_{\tilde{q}}(\mathcal{G})$ follows from Lemma 4.8.11. It is easy to see $(\tilde{f}_N)_* Y_r = \tilde{f}_r(\tilde{q})$. By Lemma 4.8.12 and 13 $\tilde{II}^\vee(\tilde{\rho}^\vee_* Y_r, \tilde{\rho}^\vee_* Y_s) =$

$-((\tilde{f}_0)_* Y_r,\ \tilde{f}_s(\tilde{q})) = -\langle(\tilde{\omega}_{s0})_{\tilde{q}},\ Y_r\rangle = \langle(\tilde{\omega}_{0s})_{\tilde{q}},\ Y_r\rangle = \sum_t \tilde{B}_{0st}(\tilde{q})\langle(\tilde{\omega}_{tN})_{\tilde{q}},\ Y_r\rangle = \tilde{B}_{0sr}(\tilde{q}) = \tilde{B}_{0rs}(\tilde{q})$.

4. We use vectors Y_r's in 3. By 1 every vector $Z \in T_{\tilde{\rho}^\vee(\tilde{q})}(L)$ can be written $Z = \sum_r x_r \tilde{f}_r(\tilde{q})$ with some $x_{m+1}, x_{m+2}, \dots, x_{N-1} \in \mathbf{R}$. For $X = \sum_r x_r Y_r \in T_{\tilde{q}}(\mathcal{G})$ we have $\tilde{\rho}^\vee_* X = Z$.

By 2 and by Lemma 4.8.12 and 13 $\tilde{II}^\vee(Z,\ \tilde{\rho}^\vee_* Y_s) = \sum_r x_r \tilde{II}^\vee(\tilde{\rho}^\vee_* Y_r,\ \tilde{\rho}^\vee_* Y_s) = \sum_r x_r \tilde{B}_{0rs}(\tilde{q}) = \sum_\alpha x_\alpha \tilde{B}_{0\alpha s}(\tilde{q})$. In particular, by Lemma 4.8.13 one knows that $\tilde{II}^\vee(Z,\ \tilde{\rho}^\vee_* Y_S) = 0$ for every S.

Assume $Z \in \mathrm{rad}_{\tilde{\pi}(\tilde{q})}(\tilde{II}^\vee)$. We have $0 = \tilde{II}^\vee(Z,\ \tilde{\rho}^\vee_* Y_\beta) = \sum_\alpha x_\alpha \tilde{B}_{0\alpha\beta}(\tilde{q})$ for every β. By Lemma 4.8.14 one has $x_{m+1} = x_{m+2} = \cdots = x_n = 0$.

Conversely assume $x_{m+1} = x_{m+2} = \cdots = x_n = 0$. We have $\tilde{II}^\vee(Z,\ \tilde{\rho}^\vee_* Y_\beta) = 0$ for every β. On the other hand, $\tilde{II}^\vee(Z,\ \tilde{\rho}^\vee_* Y_S) = 0$ for every S without any assumption. We have $Z \in \mathrm{rad}_{\tilde{\pi}(\tilde{q})}(\tilde{II}^\vee)$, since $\tilde{\rho}^\vee_* Y_s$'s are a basis of $T_{\tilde{\rho}^\vee(\tilde{q})}(L)$. We obtain 4. □

Corollary 4.10. $\mathrm{rank}_{\tilde{p}}(\tilde{II}^\vee)$ *does not depend on* $\tilde{p} \in \mathcal{E} - \lambda^{-1}(\lambda(C))$.

Theorem 4.11 (Duality of the second fundamental form. See Theorem 3.11)**.** *Let* $\tilde{p} \in \mathcal{E} - \lambda^{-1}(\lambda(C))$ *be an arbitrary point.*

1. *The tangent space* $T_{\mu(\tilde{p})}(M)$ *is an orthogonal direct sum of* $\mathrm{rad}_{\tilde{p}}(\tilde{II})$ *and the intersection* $T_{\mu(\tilde{p})}(M) \cap T_{\mu^\vee(\tilde{p})}(L)$ *of embedded tangent spaces.*
2. *The tangent space* $T_{\mu^\vee(\tilde{p})}(L)$ *is an orthogonal direct sum of* $\mathrm{rad}_{\tilde{p}}(\tilde{II}^\vee)$ *and* $T_{\mu(\tilde{p})}(M) \cap T_{\mu^\vee(\tilde{p})}(L)$.
3. *The vector space* V *has the following orthogonal direct sum decomposition:*
$$\mathbf{R}\sigma\mu(\tilde{p}) + \mathrm{rad}_{\tilde{p}}(\tilde{II}) + \big(T_{\mu(\tilde{p})}(M) \cap T_{\mu^\vee(\tilde{p})}(L)\big) + \mathrm{rad}_{\tilde{p}}(\tilde{II}^\vee) + \mathbf{R}\sigma^\vee\mu^\vee(\tilde{p})$$
4. *Let* $Z_{m+1}, Z_{m+2}, \dots, Z_n$ *be an orthogonal normal basis of* $T_{\mu(\tilde{p})}(M) \cap T_{\mu^\vee(\tilde{p})}(L)$. *The matrix* $(\tilde{II}(Z_\alpha,\ Z_\beta))$ $(m+1 \le \alpha, \beta \le n)$ *is the inverse matrix of* $(\tilde{II}^\vee(Z_\alpha,\ Z_\beta))$.

Proof. Let $\tilde{q} \in \mathcal{G}$ be a point with $\tilde{\pi}(\tilde{q}) = \tilde{p}$. By definition of $\mathcal{G}$ $\tilde{f}_1(\tilde{q}), \tilde{f}_2(\tilde{q}), \dots, \tilde{f}_n(\tilde{q})$ are an orthogonal normal basis of $T_{\mu(\tilde{p})}(M)$. By Lemma 4.9.1 $\tilde{f}_{m+1}(\tilde{q}), \tilde{f}_{m+2}(\tilde{q}), \dots, \tilde{f}_{N-1}(\tilde{q})$ are an orthogonal normal basis of $T_{\mu^\vee(\tilde{p})}(L)$. Thus $\tilde{f}_{m+1}(\tilde{q}), \tilde{f}_{m+2}(\tilde{q}), \dots, \tilde{f}_n(\tilde{q})$ are an orthogonal normal basis of $T_{\mu(\tilde{p})}(M) \cap T_{\mu^\vee(\tilde{p})}(L)$.

1. By definition of $\mathcal{G}$ $\tilde{f}_1(\tilde{q}), \tilde{f}_2(\tilde{q}), \dots, \tilde{f}_m(\tilde{q})$ are an orthogonal normal basis of $\mathrm{rad}_{\tilde{p}}(\tilde{II})$.

2. By Lemma 4.9.4 $\tilde{f}_{n+1}(\tilde{q}), \tilde{f}_{n+2}(\tilde{q}), \dots, \tilde{f}_{N-1}(\tilde{q})$ are an orthogonal normal basis of $\mathrm{rad}_{\tilde{p}}(\tilde{II}^\vee)$.

3. It follows from 1 and 2.

4. By choosing an appropriate point $\tilde{q} \in \mathcal{G}$ with $\tilde{\pi}(\tilde{q}) = \tilde{p}$ we can assume $Z_\alpha = \tilde{f}_\alpha(\tilde{q})$. By Lemma 4.8.8 we have $\tilde{II}(Z_\alpha,\ Z_\beta) = \tilde{A}_{N\alpha\beta}(\tilde{q})$. (Note that $\tilde{f}_i(\tilde{q}) = \sigma_*\tilde{\rho}_* X_i$ and $\tilde{\rho}_* X_i$ in Lemma 4.8.8 are identified.) By Lemma 4.9.3 $\tilde{II}^\vee(Z_\alpha,\ Z_\beta) = \tilde{B}_{0rs}(\tilde{q})$. Thus by Lemma 4.8.14 we have claim 4. □

Corollary 4.12. $\mathrm{rank}(\tilde{II}) = \mathrm{rank}(\tilde{II}^\vee)$.

Proof. By Corollary 4.10 it is obvious. □

The intersection $S \cap W$ or $S^\vee \cap W$ of a vector subspace W of V with S or $S^\vee$ is called a *totally geodesic* submanifold in S or $S^\vee$. When $\epsilon = +1$, a totally geodesic submanifold of S is a *great sphere*.

For a vector $a \in S^\vee$ we denote

$$R_a = \{p \in M \mid \hat{T}_p(M) \text{ is orthogonal to } a\},$$

and call R_a *the contact locus* of M in the direction a, following Wallace [15]. Similarly for a vector $b \in S$ we denote

$$P_b = \{p \in L \mid \hat{T}_p(L) \text{ is orthogonal to } b\},$$

and call P_b the contact locus of L in the direction b.

Proposition 4.13. 1. *$R_a \neq \emptyset$ if and only if $a \in M^\vee = \lambda(\mathcal{E})$.*

2. *Let a be a point in $\bar{L} = \sigma^\vee(L) = \lambda(\mathcal{E}) - \lambda(C)$. The number of connected components of R_a is finite. Every connected component of R_a is smooth and of dimension m. The restriction of the map σ to each connected component of R_a is an isomorphism to either a totally geodesic submanifold in S or one point in S.*
3. *If $a \in \bar{L}$ is a smooth point of $\bar{L}$, then R_a is either connected or a set of two points.*
4. *If $P_b \neq \emptyset$, then every connected component of P_b is smooth and of dimension $N - n - 1$, and the restriction of the map $\sigma^\vee$ to each connected component is an isomorphism to an open set of a totally geodesic submanifold in $S^\vee$.*

Proof. 1. It is obvious by the definition of $M^\vee$.
2, 3. For $a \in \sigma^\vee(L)$ by definition we have $R_a = \mu((\sigma^\vee\mu^\vee)^{-1}(a))$. By Lemma 3.2.1 $R_a \cong (\sigma^\vee\mu^\vee)^{-1}(a)$. Thus R_a is the disjoint union of $\mu(\mu^{\vee-1}(p))$ for $p \in \sigma^{\vee-1}(a)$. Note that $\sigma^{\vee-1}(a)$ is a set of only one point if and only if $a \in \bar{L}$ is a smooth point of $\bar{L}$. Our claims follow from Theorem 3.11.
4. By Theorem 3.11 one knows $P_b = \mu^\vee(F_b)$ where $F_b = ((\sigma\mu)^{-1}(b) \cup (\sigma\mu)^{-1}(-b)) \cap(\mathcal{E} - \lambda^{-1}(\lambda(C)))$. By Lemma 3.2.2 every conected component is mapped isomorphically onto its image by $\lambda = \sigma^\vee\mu^\vee$. By definition of $\mathcal{E}$ the image is an open set of a totally geodesic submanifold. □

Theorem 4.14. *Let $\tilde{p} \in \mathcal{E} - \lambda^{-1}(\lambda(C))$ be an arbitrary point.*

1. *The contact locus $R_{\sigma^\vee\mu^\vee(\tilde{p})} \subset M$ contains the point $\mu(\tilde{p})$ and $\mathrm{rad}_{\tilde{p}}(\tilde{II}) = T_{\mu(\tilde{p})}(R_{\sigma^\vee\mu^\vee(\tilde{p})})$.*
2. *The contact locus $P_{\sigma\mu(\tilde{p})} \subset L$ contains the point $\mu^\vee(\tilde{p})$ and $\mathrm{rad}_{\tilde{p}}(\tilde{II}^\vee) = T_{\mu^\vee(\tilde{p})}(P_{\sigma\mu(\tilde{p})})$.*

Proof. 1. It is easy to see $\mu(\tilde{p}) \in R_{\sigma^\vee\mu^\vee(\tilde{p})}$. Consider $\tilde{R}_{\tilde{p}} = (\sigma^\vee\mu^\vee)^{-1}(\sigma^\vee\mu^\vee(\tilde{p}))$. Since $\sigma^\vee$ is an immersion and since the rank of $\mu^\vee$ is constant around $\tilde{p}$, $\tilde{p} \in \tilde{R}_{\tilde{p}}$ is a smooth point of the analytic set $\tilde{R}_{\tilde{p}}$ and we have an exact sequence $0 \to T_{\tilde{p}}(\tilde{R}_{\tilde{p}}) \to T_{\tilde{p}}(\mathcal{E}) \xrightarrow{\mu^\vee_*} T_{\mu^\vee(\tilde{p})}(L) \to 0$. By Corollary 4.7 one knows $\dim T_{\tilde{p}}(\tilde{R}_{\tilde{p}}) = m$, since $\lambda = \sigma^\vee\mu^\vee$. Let $Z \in T_{\tilde{p}}(\tilde{R}_{\tilde{p}})$ be an arbitrary vector. Choose a point $\tilde{q} \in \mathcal{G}$ with $\tilde{\pi}(\tilde{q}) = \tilde{p}$. We have a vector $Z' \in T_{\tilde{q}}(\mathcal{G})$ with $\tilde{\pi}_*Z' = Z$. We have $(\tilde{f}_N)_*Z' = \sigma^\vee_*\mu^\vee_*\tilde{\pi}_*Z' = \mu^\vee_*Z = 0$, and $\mu_*Z = \mu_*\tilde{\pi}_*Z' = \tilde{\rho}_*Z'$. By Lemma 4.1.4 $\tilde{II}(\mu_*Z, \tilde{\rho}_*X) = \tilde{II}(\tilde{\rho}_*Z', \tilde{\rho}_*X) = -((\tilde{f}_N)_*Z', (\tilde{f}_0)_*X) = 0$ for any $X \in T_{\tilde{q}}(\mathcal{G})$. Thus $\mu_*(T_{\tilde{p}}(\tilde{R}_{\tilde{p}})) \subset \mathrm{rad}_{\tilde{p}}(\tilde{II})$. By Lemma 3.2.1 $\tilde{R}_{\tilde{p}} = (\sigma^\vee\mu^\vee)^{-1}(\sigma^\vee\mu^\vee(\tilde{p})) \cong \mu((\sigma^\vee\mu^\vee)^{-1}(\sigma^\vee\mu^\vee(\tilde{p}))) = R_{\sigma^\vee\mu^\vee(\tilde{p})}$. Thus $T_{\tilde{p}}(\tilde{R}_{\tilde{p}}) \cong \mu_*(T_{\tilde{p}}(\tilde{R}_{\tilde{p}})) = T_{\mu(\tilde{p})}(R_{\sigma^\vee\mu^\vee(\tilde{p})})$.

In particular, $\dim T_{\mu(\tilde{p})}(R_{\sigma^\vee\mu^\vee(\tilde{p})}) = \dim \mathrm{rad}_{\tilde{p}}(\tilde{II}) = m$. Thus we obtain the desired equality.
2. By Lemma 3.2.2 $\mu^\vee$ induces an isomorphism $(\sigma\mu)^{-1}(\sigma\mu(\tilde{p})) \cong \mu^\vee((\sigma\mu)^{-1}(\sigma\mu(\tilde{p})))$. Using Lemma 4.9.2 and this fact, by the similar reasoning as in 1 we obtain the desired equality. □

5. Degeneration

In this section we consider degeneration of the dual variety and degeneration of the Gauss map.

Let X be a quasi-analytic subset of a real-analytic manifold. By definition we have an analytic subset $Y \subset X$ which is dense in X. We define that the dimension of X is the dimension of Y. The dimension of X is denoted by $\dim X$. This $\dim X$ does not depend on the choice of Y. If X is analytic, then it coincides with the dimension as an analytic set.

If the dual variety $M^\vee$ of an analytic immersion $\sigma : M \to S$ has dimension less than $N-1$, then we say that $M^\vee$ is *degenerated*.

Proposition 5.1. *Let M be n-dimensional smooth connected compact real-analytic manifold and $\sigma : M \to S$ be an almost injective real-analytic immersion. Let $M^\vee$ denote the dual variety of M.*

1. $\dim M^\vee = N - n - 1 + \mathrm{rank}\,(\tilde{II})$.
2. *The dual variety $M^\vee$ is degenerated if and only if at a general point of M the second fundamental form of M in S in a general normal direction is degenerated.*
3. *In the case $\epsilon = -1$ of hyperbolic N-space, the dual variety is never degenerated.*
4. *In the case $\epsilon = +1$ of an N-sphere, if $\sigma(M) \neq \tau\sigma(M)$, then the dual variety is not degenerated, where $\tau : S \to S$ denotes the antipodal map.*

Proof. 1. By Lemma 4.6 it is obvious.
2. It follows from 1.
3, 4. It follows from Proposition 3.7. □

Let $g : M \to G(n+1, V)$ denote the Gauss map of M. We say that g *is degenerated,* if $\mathrm{rank}\,(g) < n$.

Proposition 5.2. 1. *We have a dense open set U in $g(M)$ such that for any point $\xi \in U$ the inverse image $g^{-1}(\xi)$ of a point ξ is mapped by σ isomorphically onto either a totally geodesic submanifold in S or one point.*
2. *If g is degenerated, then the dual variety $M^\vee$ of M is degenerated.*

Proof. We consider the space $\mathcal{F}$ of orthogonal normal frames on M again. We use the same notations as in previous sections. Moreover, for $n < \zeta \leq N$ we define a map $\pi_\zeta : \mathcal{F} \to \mathcal{E}$ by $\pi_\zeta(q) = (\rho(q), f_\zeta(q))$ for $q \in \mathcal{F}$. By definition $\pi = \pi_N$. Let L_0 be the set of smooth points on $\lambda(\mathcal{E}) - \lambda(C)$ and $W = \bigcap_\zeta (\lambda\pi_\zeta)^{-1}(L_0)$. By Corollary 2.32 W is dense in $\mathcal{F}$. Let $U = \{\xi \in g(M) \mid (g\rho)^{-1}(\xi) \cap W \neq \emptyset\}$. It is easy to see that U is open and dense in $g(M)$. Let $\xi \in U$ be an arbitrary point. By definition we have a point $q \in W$ with $g\rho(q) = \xi$. We denote $\tilde{p}_\zeta = \pi_\zeta(q) \in \mathcal{E}$ and $p = \rho(q) = \mu(\tilde{p}_{n+1}) = \mu(\tilde{p}_{n+2}) = \cdots = \mu(\tilde{p}_N) \in M$. By the definition of W we have $\lambda(\tilde{p}_\zeta) \in L_0$ for every ζ. Thus by Proposition 4.13.2 and 3 for every ζ the contact locus $R_{\lambda(\tilde{p}_\zeta)}$ is mapped by σ isomorphically onto either a totally geodesic submanifold in S or one point. Now, by the definition of π_ζ's we have $\hat{T}_p(M) = \{\lambda(\tilde{p}_{n+1}), \lambda(\tilde{p}_{n+2}), \ldots, \lambda(\tilde{p}_N)\}^\perp$. Thus $g^{-1}(\xi) = \bigcap_\zeta R_{\lambda(\tilde{p}_\zeta)}$, and we obtain claim 1, since the intersection of some totally geodesic submanifolds is again totally geodesic. If $\dim g^{-1}(\xi) > 0$, then $\dim \mathrm{rad}_{\tilde{p}_N}(\tilde{II}) = \dim R_{\lambda(\tilde{p}_N)} \geq \dim g^{-1}(\xi) > 0$ by Proposition 4.13.2 and Lemma 2.24.1. Thus by Proposition 5.1.1 $M^\vee$ is degenerated. □

Corollary 5.3. 1. *In the case $\epsilon = -1$ of hyperbolic N-space, the Gauss map g is never degenerated.*
2. *In the case $\epsilon = +1$ of an N-sphere, if $\sigma(M) \neq \tau\sigma(M)$, then the Gauss map g is not degenerated.*

In the remainder of this section we show the theorem below. (See Griffiths, Harris [6, (2.29) on page 393].)

Theorem 5.4. *We consider the case $\epsilon = +1$ of an N-sphere. Let M be an n-dimensional smooth connected compact real-analytic manifold and $\sigma : M \to S$ be an almost injective real-analytic immersion. Assume that the Gauss map $g : M \to G(n+1, V)$ is degenerated. Then $\sigma(M) = \tau\sigma(M)$ and $\dim g(M)$ is an even integer, where $\tau : S \to S$ denotes the antipodal map.*

Corollary 5.5. *We consider the same situation as in Theorem 5.4. Assume $n = 1$ or $n = 2$. Then, the Gauss map g is degenerated if and only if σ is an embedding and $\sigma(M)$ is a totally geodesic submanifold in S.*

We write $l = n - \mathrm{rank}\,(g)$. In this section we use the ranges of indices

$$0 \leq E, F, G \leq N;$$
$$1 \leq i, j, k \leq n; \quad n+1 \leq \zeta, \eta, \theta \leq N;$$
$$1 \leq e, f, g \leq l; \quad l+1 \leq a, b, c \leq n.$$

For $p \in M$ we denote $T_p(M/G(n+1, V)) = \mathrm{Ker}\,\big(g_* : T_p(M) \to T_{g(p)}(G(n+1, V))\big)$. Let $U \subset g(M)$ be the open dense subset in Proposition 5.2. We here introduce the manifold $\mathcal{H}$ and several maps. Put

$$\begin{aligned}\mathcal{H} =\{ &q \in \mathcal{F} \mid \rho(q) \in g^{-1}(U),\\ &f_1(q), f_2(q), \ldots, f_l(q) \text{ are a basis of } T_{\rho(q)}(M/G(n+1, V))\}.\end{aligned}$$

Let $\bar{\iota} : \mathcal{H} \to \mathcal{F}$ denote the inclusion map. We denote $\bar{f}_E = f_E\bar{\iota}$, $\bar{\rho} = \rho\bar{\iota}$, and $\bar{A}_{\zeta ij} = A_{\zeta ij}\bar{\iota}$. The map $\bar{\rho} : \mathcal{H} \to g^{-1}(U)$ is the projection of a principal $O(l) \times O(n-l) \times O(N-n)$-bundle. Thus $\mathcal{H}$ is a smooth analytic manifold, and we can consider the Maurer-Cartan forms $\bar{\omega} = \bar{\iota}^*\omega$, $\bar{\omega}_{EF} = \bar{\iota}^*\omega_{EF}$ on $\mathcal{H}$.

We obtain the commutative diagram below.

$$\begin{array}{ccccc} & & \mathcal{H} & = & \mathcal{H} \\ & & \downarrow \bar{\rho} & & \downarrow \bar{f}_0 \\ G(n+1, V) & \xleftarrow{g} & g^{-1}(U) & \xrightarrow{\sigma} & V \subset V \end{array}$$

Let $\bar{q} \in \mathcal{H}$ be an arbitrarily fixed point.

Lemma 5.6. 1. *There is a canonical isomorphism $T_{g\bar{\rho}(\bar{q})}(G(n+1, V)) \cong \mathrm{Hom}(\hat{T}_{\bar{\rho}(\bar{q})}(M), N_{\bar{\rho}(\bar{q})}(M/S))$.*
2. *Vectors $\bar{f}_0(\bar{q}), \bar{f}_1(\bar{q}), \ldots, \bar{f}_n(\bar{q})$ are a basis of $\hat{T}_{\bar{\rho}(\bar{q})}(M)$, and vectors $\bar{f}_\zeta(\bar{q})$'s are a basis of $N_{\bar{\rho}(\bar{q})}(M/S)$.*

3. *There are vectors* $X_1, X_2, \ldots, X_n \in T_{\bar q}(\mathcal{H})$ *with* $\langle(\bar\omega_{i0})_{\bar q},\, X_i\rangle = 1$ *and* $\langle(\bar\omega_{i0})_{\bar q},\, X_j\rangle = 0$ *for* $i \neq j$. *Vectors* $\bar\rho_* X_1, \bar\rho_* X_2, \ldots, \bar\rho_* X_n$ *are a basis of* $T_{\bar\rho(\bar q)}(M)$. *Vectors* $\bar\rho_* X_1, \bar\rho_* X_2, \ldots, \bar\rho_* X_l$ *are a basis of* $T_{\bar\rho(\bar q)}(M/G(n+1,\, V))$.
4. *We have* $(g_*\bar\rho_* X_i)(x_0\bar f_0(\bar q) + x_1\bar f_1(\bar q) + \cdots + x_n\bar f_n(\bar q)) = \sum_{\zeta,j} x_j\bar A_{\zeta ji}(\bar q)\bar f_\zeta(\bar q)$ *for* $g_* : T_{\bar\rho(\bar q)}(M) \to \mathrm{Hom}(\hat T_{\bar\rho(\bar q)}(M),\, N_{\bar\rho(\bar q)}(M/S))$.
5. $\bar A_{\zeta ie} = 0$ *and* $\bar\omega_{\zeta e} = 0$.

Proof. 1, 2, 3. Trivial.
4. It is not difficult to show $(g_*\bar\rho_* X)(x_0\bar f_0(\bar q) + x_1\bar f_1(\bar q) + \cdots + x_n\bar f_n(\bar q)) = \sum_{i,\zeta} x_i\langle(\bar\omega_{\zeta i})_{\bar q},\, X\rangle\bar f_\zeta(\bar q)$ for any $X \in T_{\bar q}(\mathcal{H})$ and for any real numbers $x_0, x_1, \ldots, x_n$. Recall $\bar\omega_{\zeta i} = \sum_j \bar A_{\zeta ij}\bar\omega_{j0}$ and $\bar A_{\zeta ij} = \bar A_{\zeta ji}$ by Lemma 4.2. The equality in 4 follows from these equalities.
5. Since $\bar\rho_* X_e \in T_{\bar\rho(\bar q)}(M/G(n+1,\, V))$ for $1 \le e \le l$, we have $0 = \langle(\bar\omega_{\zeta i})_{\bar q},\, X_e\rangle = \bar A_{\zeta ie}(\bar q)$ for any ζ and i. Since $\bar q \in \mathcal{H}$ is an arbitrary point, we have $\bar A_{\zeta ei} = \bar A_{\zeta ie} = 0$. Thus $\bar\omega_{\zeta e} = \sum_i \bar A_{\zeta ei}\bar\omega_{i0} = 0$. □

Lemma 5.7. 1. *If real numbers* $x_{l+1}, x_{l+2}, \ldots, x_n$ *satisfy* $\sum_b \bar A_{\zeta ab}(\bar q)x_b = 0$ *for any* ζ *and for any* a, *then* $x_{l+1} = x_{l+2} = \cdots = x_n = 0$.
2. $(N-n)\times(n-l)$ *of vectors* $(\bar A_{\zeta a,l+1}(\bar q), \bar A_{\zeta a,l+2}(\bar q), \ldots, \bar A_{\zeta an}(\bar q)) \in \mathbf{R}^{n-l}$ *span* $\mathbf{R}^{n-l}$.
3. *There are functions* $\bar C_{\zeta ab}$ *on a neighborhood of* $\bar q \in \mathcal{H}$ *with* $\sum_{\zeta,a}\bar C_{\zeta ab}\bar A_{\zeta ab} = 1$ *and* $\sum_{\zeta,a}\bar C_{\zeta ab}\bar A_{\zeta ac} = 0$ *for* $b \neq c$.

Proof. 1. By Lemma 5.6 one knows the linear map $(x_{l+1}, x_{l+2}, \ldots, x_n) \mapsto \sum_b \bar A_{\zeta ab}(\bar q)x_b$ corresponds to the injective map $T_{\bar\rho(\bar q)}(M)/T_{\bar\rho(\bar q)}(M/G(n+1,\, V)) \to T_{g\bar\rho(\bar q)}(G(n+1,\, V))$. Thus 1 follows.
2. Considering the dual linear map of the one in 1, we obtain 2.
3. It follows from 2. □

Lemma 5.8. *There is a unique set of functions* $\bar D_{eab}$ *on* $\mathcal{H}$ *with* $\bar\omega_{ae} = \sum_b \bar D_{eab}\bar\omega_{b0}$ *for every* a, e.

Remark. They do not necessarily satisfy $\bar D_{eab} = \bar D_{eba}$.

Proof. The uniqueness follows from that $\bar\omega_{b0}$'s are linearly independent at every point on $\mathcal{H}$. We will show the existence. Because of the uniqueness we can show it locally on $\mathcal{H}$. We have $\bar\omega_{a0} = 0$ by Lemma 3.5.4 and $\bar\omega_{\zeta e} = 0$ by Lemma 5.6.5. By Lemma 4.2 $\bar\omega_{\zeta a} = \sum_b \bar A_{\zeta ab}\bar\omega_{b0}$. Thus $0 = -d\,\bar\omega_{\zeta e} = \sum_a \bar\omega_{\zeta a}\wedge\bar\omega_{ae} = \sum_{a,b}\bar A_{\zeta ab}\bar\omega_{b0}\wedge\bar\omega_{ae} = \sum_b \bar\omega_{b0}\wedge(\sum_a \bar A_{\zeta ba}\bar\omega_{ae})$. One knows that we have functions $\bar D_{\zeta eab}$ on $\mathcal{H}$ with $\sum_a \bar A_{\zeta ba}\bar\omega_{ae} = \sum_a \bar D_{\zeta eba}\bar\omega_{a0}$. We can check that functions $\bar D_{eab} = \sum_{\zeta,c}\bar C_{\zeta ca}\bar D_{\zeta ecb}$ on a neighborhood of $\bar q$ satisfy the desired condition. □

Proof of Theorem 5.4. Assume that g is degenerated. We have $l \ge 1$. The equality $\sigma(M) = \tau\sigma(M)$ follows from Corollary 5.3.2.

Assume moreover that $n - l = \dim g(M)$ is an odd integer. We will deduce a contradiction.

Let $\bar D_{eab}$ be the functions on $\mathcal{H}$ in Lemma 5.8. By $D = (\sum_e \bar D_{eab})$ we denote the $(n-l)\times(n-l)$-matrix of functions on $\mathcal{H}$, and by E we denote the unit matrix of degree $n-l$. Let

$$H = \{(t,\, u,\, \bar q) \in \mathbf{R}\times\mathbf{R}\times\mathcal{H} \mid \det(tE + uD(\bar q)) = 0, t^2 + (n-l)u^2 = 1\}.$$

We define maps $\alpha : H \to \mathbf{R}$, $\beta : H \to \mathbf{R}$ and $\gamma : H \to \mathcal{H}$ by $\alpha(t, u, \bar{q}) = t$, $\beta(t, u, \bar{q}) = u$ and $\gamma(t, u, \bar{q}) = \bar{q}$. H is an analytic set in $\mathbf{R} \times \mathbf{R} \times \mathcal{H}$. Since $\det(tI + uD(\bar{q}))$ is a polynomial with real coefficients of degree $n - l$, and since $n - l$ is odd, for every $\bar{q} \in \mathcal{H}$ and for every non-zero real number u, there exists at least one real number t with $\det(tI + uD(\bar{q})) = 0$. Putting $t' = t/\sqrt{t^2 + (n-l)u^2}$ and $u' = u/\sqrt{t^2 + (n-l)u^2}$, we have $(t', u', \bar{q}) \in H$. Thus γ is surjective. Since $\gamma^{-1}(\bar{q})$ is a finite set for every $\bar{q} \in \mathcal{H}$, one knows $\dim H = \dim \mathcal{H}$. Thus we have a non-empty smooth open set $H' \subset H$ such that $\gamma|H' : H' \to \mathcal{H}$ is an analytic isomorphism onto its image. Also α and β are analytic on H'.

Let $q_0 = (t_0, u_0, \bar{q}_0) \in H'$ be a point, and $X_1, X_2, \ldots, X_n \in T_{q_0}(H')$ be tangent vectors with $\langle (\gamma^* \bar{\omega}_{i0})_{q_0}, X_i \rangle = 1$ and $\langle (\gamma^* \bar{\omega}_{i0})_{q_0}, X_j \rangle = 0$ for $i \neq j$. Such vectors $X_1, X_2, \ldots, X_n$ exist, since $\gamma|H'$ is an isomorphism. On the other hand, since $\det(t_0 I + u_0 D(\bar{q}_0)) = 0$, we have a non-zero vector $x \in \mathbf{R}^{n-l}$ with $(t_0 I + u_0 D(\bar{q}_0))x = 0$. Writing $x = {}^t(x_{l+1}, x_{l+2}, \ldots, x_n)$ with $x_a \in \mathbf{R}$, we put $X = \sum_a x_a X_a \in T_{q_0}(H')$. We have $\sigma_* \bar{\rho}_* \gamma_* X = (\bar{f}_0)_* \gamma_* X = \sum_a x_a \bar{f}_a(\bar{q}_0) \notin T_{\bar{\rho}(\bar{q}_0)}(M/G(n+1, V))$, since $x \neq 0$. Thus $g_* \bar{\rho}_* \gamma_* X \neq 0$.

We here consider the map $\bar{f} = \alpha(\gamma^* \bar{f}_0) + \beta \sum_e (\gamma^* \bar{f}_e) : H' \to V$. Note that by Proposition 5.2 $\delta = (g, \sigma) : g^{-1}(U) \to G(n+1, V) \times V$ is an isomorphism onto its image and $\sigma(g^{-1}(g\bar{\rho}(\bar{q}))) = \{y_0 \bar{f}_0(\bar{q}) + y_1 \bar{f}_1(\bar{q}) + \cdots + y_l \bar{f}_l(\bar{q}) \mid y_0, y_1, \ldots, y_l \in \mathbf{R}, y_0^2 + y_1^2 + \cdots + y_l^2 = 1\}$ for every $\bar{q} \in \mathcal{H}$. Since $\alpha^2 + (n-l)\beta^2 = 1$ by definition, one knows that $\bar{f}(q') \in \sigma(g^{-1}(g\bar{\rho}\gamma(q')))$ for every $q' \in H'$. Thus the image of the map $(g\bar{\rho}\gamma, \bar{f}) : H' \to G(n+1, V) \times V$ is contained in the image of $\delta = (g, \sigma)$. We have a map $\phi = \delta^{-1}(g\bar{\rho}\gamma, \bar{f}) : H' \to g^{-1}(U)$ with $g\phi = g\bar{\rho}\gamma$ and $\sigma\phi = \bar{f}$. Thus $g_* \phi_* X = g_* \bar{\rho}_* \gamma_* X \neq 0$, and $\phi_* X \notin T_{\phi(q_0)}(M/G(n+1, V))$. Besides, since $\bar{f}_* X = \sigma_* \phi_* X$ is orthogonal to $\bar{f}(q_0)$, $\bar{f}_* X$ is not contained in the linear space spanned by $\bar{f}(q_0) = \sigma\phi(q_0)$ and $T_{\phi(q_0)}(M/G(n+1, V))$.

Let $\xi_0 = g\phi(q_0) = g\bar{\rho}(\bar{q}_0) \in U$ and $F = g^{-1}(\xi_0)$. F contains points $\phi(q_0)$ and $\bar{\rho}(\bar{q}_0)$. By Proposition 5.2 $\sigma(F)$ is a totally geodesic submanifold in S. By definition, the linear space spanned by $\bar{f}(q_0)$ and $T_{\phi(q_0)}(M/G(n+1, V))$ coincides with $\hat{T}_{\phi(q_0)}(F)$. Since $\sigma(F)$ is totally geodesic, we have $\hat{T}_{\phi(q_0)}(F) = \hat{T}_{\bar{\rho}(\bar{q}_0)}(F)$. By definition, $\hat{T}_{\bar{\rho}(\bar{q}_0)}(F)$ is the linear space spanned by $\bar{f}_0(\bar{q}_0), \bar{f}_1(\bar{q}_0), \ldots, \bar{f}_l(\bar{q}_0)$.

One concludes that $\bar{f}_* X$ is *not* a linear combination of $\bar{f}_0(\bar{q}_0), \bar{f}_1(\bar{q}_0), \ldots, \bar{f}_l(\bar{q}_0)$.

We here compute $\bar{f}_* X$. By Lemma 4.8.2, 3, and 5 and by Lemma 5.6.5 we have

$$\begin{aligned}
\bar{f}_* X &= u_0 \sum_e \langle (\gamma^* \bar{\omega}_{0e})_{q_0}, X \rangle \bar{f}_0(\bar{q}_0) \\
&\quad + \sum_f \left\{ t_0 \langle (\gamma^* \bar{\omega}_{f0})_{q_0}, X \rangle + u_0 \sum_e \langle (\gamma^* \bar{\omega}_{fe})_{q_0}, X \rangle \right\} \bar{f}_f(\bar{q}_0) \\
&\quad + \sum_a \left\{ t_0 \langle (\gamma^* \bar{\omega}_{a0})_{q_0}, X \rangle + u_0 \sum_e \langle (\gamma^* \bar{\omega}_{ae})_{q_0}, X \rangle \right\} \bar{f}_a(\bar{q}_0).
\end{aligned}$$

Now, for each a we have

$$\begin{aligned}
& t_0 \langle (\gamma^* \bar{\omega}_{a0})_{q_0}, X \rangle + u_0 \sum_e \langle (\gamma^* \bar{\omega}_{ae})_{q_0}, X \rangle \\
=\ & t_0 \langle (\gamma^* \bar{\omega}_{a0})_{q_0}, X \rangle + u_0 \sum_b \sum_e \bar{D}_{eab}(\bar{q}_0) \langle (\gamma^* \bar{\omega}_{b0})_{q_0}, X \rangle \\
=\ & t_0 x_a + u_0 \sum_b \sum_e \bar{D}_{eab}(\bar{q}_0) x_b \\
=\ & 0
\end{aligned}$$

Thus $\bar{f}_* X$ is a linear combination of $\bar{f}_0(\bar{q}_0), \bar{f}_1(\bar{q}_0), \ldots, \bar{f}_l(\bar{q}_0)$, which is a contradiction. □

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