SOME FUNDAMENTAL PROBLEMS ON REAL-ANALYTIC SETS

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1. Introduction

The category of real-analytic sets and real-analytic maps is the most important category in application. However, in spite of efforts by F. Bruhat, H. Cartan, H. Whitney et al., the basic theory of real-analytic category does not yet seem to be well-developed. In this article I would like to point out several basic problems. (Urabe [6].)

Let Q denote a *connected* real-analytic *smooth* manifold, and $X \subset Q$ a subset of Q.

2. When is X real-analytic?

We have two kinds of definitions.

Definition 2.1. We say that X is real-analytic, if for every point $x \in X$ there are an open set U with $x \in U \subset Q$ and a finite number of real-analytic functions $f_1, f_2, \dots, f_k$ on U such that $X \cap U$ coincides with the set of common zero-points of $f_1, f_2, \dots, f_k$.

Under Definition 2.1 X is not necessarily a closed subset of Q. It is locally closed, or there is an open subset U of Q containing X such that X is a closed subset of U.

Definition 2.2. We say that X is real-analytic, if for every point $x \in Q$ there are an open set U with $x \in U \subset Q$ and a finite number of real-analytic functions $f_1, f_2, \dots, f_k$ on U such that $X \cap U$ coincides with the set of common zero-points of $f_1, f_2, \dots, f_k$.

Note that only the phrase "$x \in X$" has been replaced by "$x \in Q$". Under Definition 2.2 X is closed in Q.

I myself prefer former Definition 2.1, because it gives us flexibility of the theory and it reduces non-essential descriptions in related subjects, in which real-analytic sets in Definition 2.1 naturally appear.

3. When is a point $x \in X$ smooth?

Definition 3.1. A real-analytic germ (X, x) of sets is *smooth*, if (X, x) is real-analytically isomorphic to $(\mathbf{R}^d, 0)$ where $d = \dim(X, x)$ and $0 \in \mathbf{R}^d$ is a point.

Example 3.2. Let $k = 4$ or 5, $X = \{(x, y) \in \mathbf{R}^2 \mid x^3 = y^k\}$ and $0 = (0, 0)$. Then, the germ $(X, 0)$ is *not* smooth under above Definition 3.1. However, $(X, 0)$ is isomorphic to $(\mathbf{R}, 0)$ in the category of germs of sets and maps of class C^1. Note that $|y|^{4/3}$ and $y|y|^{2/3}$ are C^1-functions on $\mathbf{R}$.

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4. When can we define the global concept of irreducible decompositions?

Definition 4.1. We say that a real-analytic subset $X \subset Q$ is *global*, if we have a set F of real-analytic functions defined globally on Q such that X coincides with the set of common zero-points of members in F.

Definition 4.2. We say that a global real-analytic subset $X \subset Q$ is *reducible*, if $X = Y \cup Z$ for some global real-analytic subsets $Y, Z \subset Q$ with $X \neq Y$ and $X \neq Z$. We say that X is *irreducible*, if it is not reducible.

Theorem 4.3. *Any global irreducible real-analytic subset in Q is connected.*

Definition 4.4. Let X be a global real-analytic subset of Q. A family S of global irreducible real-analytic subsets in Q is called the *irreducible decomposition* of X, if the union of members in S coincides with X and if any two members in S have no inclusion relation.

In *compact* case we can show the following:

Theorem 4.5. *Any compact global real-analytic set X in Q has a unique irreducible decomposition.*

Example 4.6. Consider an *irreducible* polynomial $f(x, y) = y^2 - x^2(x - 1)$ for example. The set $X = \{(x, y) \in \mathbf{R}^2 \mid f(x, y) = 0\}$ of zero-points of $f(x, y)$ is a global real-analytic set. Drawing the picture of X, we see that it is not connected. Thus, by Theorem 4.3 X is reducible.

Example 4.7. Let m be a positive integer and $X_m = \{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 = m^2\}$. The union $X = \bigcup_{m=1}^{\infty} X_m$ is a non-compact global real-analytic subset with countably many connected components.

Problem 4.8. Give generalization of Theorem 4.5 for real-analytic subsets which are not necessarily closed.

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